Supersymmetric black holes and the attractor mechanism in 4-dimensional sugras

Tomás Ortín

(I.F.T. UAM/CSIC, Madrid)

Talk given on the 3rd of June 2010 at the *III Miniworkshop on String Theory 2010, Universidad de Oviedo*

Work done in collaboration with *P. Meessen* (University of Oviedo) and *S. Vaulà* (IFT UAM/CSIC, Madrid)

Plan of the Talk:

1 – Introduction: the search for all 4-d susy solutions

1 – Introduction: the search for all 4-d susy solutions

☞ Gibbons & Hull (1982) (Pure $N = 2$ supergravity).

1 – Introduction: the search for all 4-d susy solutions

- ☞ Gibbons & Hull (1982) (Pure $N = 2$ supergravity).
- ☞ Tod (1983) (Pure $N = 2$ supergravity).

1 – Introduction: the search for all 4-d susy solutions

- ☞ Gibbons & Hull (1982) (Pure $N = 2$ supergravity).
- ☞ Tod (1983) (Pure $N = 2$ supergravity). $\Rightarrow$ A complete answer is possible.

1 – Introduction: the search for all 4-d susy solutions

- ☞ Gibbons & Hull (1982) (Pure $N = 2$ supergravity).
- ☞ Tod (1983) (Pure $N = 2$ supergravity). ⇒ A complete answer is possible.
- ☞ Tod (1995) (Pure $N = 4$ supergravity).

1 – Introduction: the search for all 4-d susy solutions

- Gibbons & Hull (1982) (Pure $N = 2$ supergravity).
- Tod (1983) (Pure $N = 2$ supergravity). $\Rightarrow$ A complete answer is possible.
- Tod (1995) (Pure $N = 4$ supergravity).
- Gauntlett, Gutowski, Hull, Pakis & Reall (2002) (Pure $N = 1$ $d = 5$ supergravity).

Spinor-bilinears method

1 – Introduction: the search for all 4-d susy solutions

- Gibbons & Hull (1982) (Pure $N = 2$ supergravity).
- Tod (1983) (Pure $N = 2$ supergravity). $\Rightarrow$ A complete answer is possible.
- Tod (1995) (Pure $N = 4$ supergravity).
- Gauntlett, Gutowski, Hull, Pakis & Reall (2002) (Pure $N = 1$ $d = 5$ supergravity).

Spinor-bilinears method

- **2003**: Gauntlett & Pakis + Gauntlett, Gutowski & Pakis ($N = 1$ $d = 11$); Gauntlett & Gutowski (Gauged $N = 1$ $d = 5$); Caldarelli & Klemm (Pure gauged $N = 2$ $d = 4$); Gutowski, Martelli & Reall; Chamseddine, Figueroa-O'Farrill & Sabra ($N = (2, 0)$ $d = 6$)

1 – Introduction: the search for all 4-d susy solutions

- ☞ Gibbons & Hull (1982) (Pure $N = 2$ supergravity).
- ☞ Tod (1983) (Pure $N = 2$ supergravity). ⇒ A complete answer is possible.
- ☞ Tod (1995) (Pure $N = 4$ supergravity).
- ☞ Gauntlett, Gutowski, Hull, Pakis & Reall (2002) (Pure $N = 1$ $d = 5$ supergravity).

Spinor-bilinears method

- ☞ **2003**: Gauntlett & Pakis + Gauntlett, Gutowski & Pakis ($N = 1$ $d = 11$); Gauntlett & Gutowski (Gauged $N = 1$ $d = 5$); Caldarelli & Klemm (Pure gauged $N = 2$ $d = 4$); Gutowski, Martelli & Reall; Chamseddine, Figueroa-O'Farrill & Sabra ($N = (2, 0)$ $d = 6$)
- ☞ **2004**: Cariglia & Mac Conamhna ($N = 1$ $d = 7$ and gauged $N = (2, 0)$ $d = 6$)

1 – Introduction: the search for all 4-d susy solutions

- Gibbons & Hull (1982) (Pure $N = 2$ supergravity).
- Tod (1983) (Pure $N = 2$ supergravity). $\Rightarrow$ A complete answer is possible.
- Tod (1995) (Pure $N = 4$ supergravity).
- Gauntlett, Gutowski, Hull, Pakis & Reall (2002) (Pure $N = 1$ $d = 5$ supergravity).

Spinor-bilinears method

- **2003**: Gauntlett & Pakis + Gauntlett, Gutowski & Pakis ($N = 1$ $d = 11$); Gauntlett & Gutowski (Gauged $N = 1$ $d = 5$); Caldarelli & Klemm (Pure gauged $N = 2$ $d = 4$); Gutowski, Martelli & Reall; Chamseddine, Figueroa-O'Farrill & Sabra ($N = (2, 0)$ $d = 6$)
- **2004**: Cariglia & Mac Conamhna ($N = 1$ $d = 7$ and gauged $N = (2, 0)$ $d = 6$)
- **2005**: Bellorín & O. (Pure $N = 4$ $d = 4$ revisited)

1 – Introduction: the search for all 4-d susy solutions

- Gibbons & Hull (1982) (Pure $N = 2$ supergravity).
- Tod (1983) (Pure $N = 2$ supergravity). $\Rightarrow$ A complete answer is possible.
- Tod (1995) (Pure $N = 4$ supergravity).
- Gauntlett, Gutowski, Hull, Pakis & Reall (2002) (Pure $N = 1$ $d = 5$ supergravity).

Spinor-bilinears method

- **2003**: Gauntlett & Pakis + Gauntlett, Gutowski & Pakis ($N = 1$ $d = 11$); Gauntlett & Gutowski (Gauged $N = 1$ $d = 5$); Caldarelli & Klemm (Pure gauged $N = 2$ $d = 4$); Gutowski, Martelli & Reall; Chamseddine, Figueroa-O'Farrill & Sabra ($N = (2, 0)$ $d = 6$)
- **2004**: Cariglia & Mac Conamhna ($N = 1$ $d = 7$ and gauged $N = (2, 0)$ $d = 6$)
- **2005**: Bellorín & O. (Pure $N = 4$ $d = 4$ revisited)
- **2006**: Bellorín, Meessen & O. ($N = 1$ $d = 5$ with vector multiplets); Meessen & O. ($N = 2$ $d = 4$ with vector multiplets); Hübscher, Meessen & O. ($N = 2$ $d = 4$ with vector multiplets and hypermultiplets).

☞ **2007**: Bellorín & O. (Gauged $N = 1 \quad d = 5$ with vector multiplets and hypermultiplets).

- ☞ **2007**: Bellorín & O. (Gauged $N = 1$ $d = 5$ with vector multiplets and hypermultiplets).
- ☞ **2008**: Cacciatori, Klemm, Mansi & Zorzan (Gauged $N = 1$ $d = 5$ with vector multiplets); Hübscher, Meessen, O. & Vaulà (non-Abelian Gauged $N = 2$ $d = 4$ with vector multiplets); Bellorín (Gauged $N = 1$ $d = 5$ with vector and tensor multiplets).

☞ **2007**: Bellorín & O. (Gauged $N = 1$ $d = 5$ with vector multiplets and hypermultiplets).

☞ **2008**: Cacciatori, Klemm, Mansi & Zorzan (Gauged $N = 1$ $d = 5$ with vector multiplets); Hübscher, Meessen, O. & Vaulà (non-Abelian Gauged $N = 2$ $d = 4$ with vector multiplets); Bellorín (Gauged $N = 1$ $d = 5$ with vector and tensor multiplets).

☞ **2010**: Deger, Samtleben & Sarioglu (Gauged $N = 8$ $d = 3$).

- ☞ **2007**: Bellorín & O. (Gauged $N = 1$ $d = 5$ with vector multiplets and hypermultiplets).
- ☞ **2008**: Cacciatori, Klemm, Mansi & Zorzan (Gauged $N = 1$ $d = 5$ with vector multiplets); Hübscher, Meessen, O. & Vaulà (non-Abelian Gauged $N = 2$ $d = 4$ with vector multiplets); Bellorín (Gauged $N = 1$ $d = 5$ with vector and tensor multiplets).
- ☞ **2010**: Deger, Samtleben & Sarioglu (Gauged $N = 8$ $d = 3$).

However, in $d = 4$ the spinor -bilinears method has not given satisfactory results fo $N > 2$. (It has not been tried for $d > 4$).

- ☞ **2007**: Bellorín & O. (Gauged $N = 1$ $d = 5$ with vector multiplets and hypermultiplets).
- ☞ **2008**: Cacciatori, Klemm, Mansi & Zorzan (Gauged $N = 1$ $d = 5$ with vector multiplets); Hübscher, Meessen, O. & Vaulà (non-Abelian Gauged $N = 2$ $d = 4$ with vector multiplets); Bellorín (Gauged $N = 1$ $d = 5$ with vector and tensor multiplets).
- ☞ **2010**: Deger, Samtleben & Sarioglu (Gauged $N = 8$ $d = 3$).

However, in $d = 4$ the spinor -bilinears method has not given satisfactory results fo $N > 2$. (It has not been tried for $d > 4$).

> For $N > 2$ there are too many spinor bilinears and we do not know how to extract the (**not** spacetime-geometric) information they must surely contain.

Other methods

Other methods

☞ Spinorial geometry **2004**: Gillard, Gran & Papadopoulos.

Gives a more detailed classification of supersymmetric backgrounds, but it is less useful to give general classes of solutions.

Other methods

☞ Spinorial geometry **2004**: Gillard, Gran & Papadopoulos.

Gives a more detailed classification of supersymmetric backgrounds, but it is less useful to give general classes of solutions.

☞ Timelike dimensional reduction to $d = 3$ **2009**: Bossard, Nicolai & Stelle.

Powerful, but only developed for particular classes of timelike solutions. It is difficult to recover the 4-dimensional form.

Other methods

☞ Spinorial geometry **2004**: Gillard, Gran & Papadopoulos.

Gives a more detailed classification of supersymmetric backgrounds, but it is less useful to give general classes of solutions.

☞ Timelike dimensional reduction to $d = 3$ **2009**: Bossard, Nicolai & Stelle.

Powerful, but only developed for particular classes of timelike solutions. It is difficult to recover the 4-dimensional form.

☞ Black-hole attractors **1996**: Ferrara, Kallosh & Strominger.

This mechanism can be used as a powerful tool to find partial information about extremal (supersymmetric and non-supersymmetric) black holes.

These methods give complementary information.

However, in our opinion, the spinor-bilinear method would give the most if we could solve its problems for $N > 2$.

In this talk we are going to show how to solve those problems and determine the form of **all** the timelike supersymmetric solutions of all $d = 4$ supergravities using the **spinor-bilinear method**.

2 – Review of the N=2 case

Since the timelike supersymmetric solutions of $N > 2$ turn out to be related to those of $N = 2$ theories (Hübscher, Meessen & O. (2006)), we briefly review them first.

2 – Review of the N=2 case

Since the timelike supersymmetric solutions of $N > 2$ turn out to be related to those of $N = 2$ theories (Hübscher, Meessen & O. (2006)), we briefly review them first.

The $N = 2$ supergravity multiplet is

$$\left\{e^{a}{}_{\mu}, \psi_{I\,\mu}, A^{IJ}{}_{\mu}\right\}\,, \quad I, J, \cdots = 1, 2\,, \quad \Rightarrow A^{IJ}{}_{\mu} = A^{0}{}_{\mu}\varepsilon^{IJ}\,.$$

2 – Review of the N=2 case

Since the timelike supersymmetric solutions of $N>2$ turn out to be related to those of $N=2$ theories (Hübscher, Meessen & O. (2006)), we briefly review them first.

The $N=2$ supergravity multiplet is

$$\left\{e^{a}{}_{\mu}, \psi_{I\,\mu}, A^{IJ}{}_{\mu}\right\}\,, \quad I,J,\cdots=1,2\,, \quad \Rightarrow A^{IJ}{}_{\mu}=A^{0}{}_{\mu}\varepsilon^{IJ}\,.$$

The (n) $N=2$ vector multiplets are

$$\left\{A^{i}{}_{\mu}, \lambda^{i}{}_{I}, Z^{i}\right\}\,, \quad i=1,\cdots,n\,, \quad \Rightarrow A^{\Lambda}{}_{\mu}\,, \quad \Lambda=0,\cdots,n\,.$$

2 – Review of the N=2 case

Since the timelike supersymmetric solutions of $N > 2$ turn out to be related to those of $N = 2$ theories (Hübscher, Meessen & O. (2006)), we briefly review them first.

The $N = 2$ supergravity multiplet is

$$\{e^a{}_\mu, \psi_{I\,\mu}, A^{IJ}{}_\mu\}\,, \quad I, J, \cdots = 1, 2\,, \quad \Rightarrow A^{IJ}{}_\mu = A^0{}_\mu \varepsilon^{IJ}\,.$$

The (n) $N = 2$ vector multiplets are

$$\{A^i{}_\mu, \lambda^i{}_I, Z^i\}\,, \quad i = 1, \cdots, n\,, \quad \Rightarrow A^\Lambda{}_\mu\,, \quad \Lambda = 0, \cdots, n\,.$$

The (m) hypermultiplets are

$$\{\zeta_\alpha, q^u\}\,, \quad u = 1, \cdots, 4m\,, \quad \alpha = 1, \cdots, 2m\,.$$

2 – Review of the N=2 case

Since the timelike supersymmetric solutions of $N > 2$ turn out to be related to those of $N = 2$ theories (Hübscher, Meessen & O. (2006)), we briefly review them first.

The $N = 2$ supergravity multiplet is

$$\{e^a{}_\mu, \psi_{I\,\mu}, A^{IJ}{}_\mu\}\,, \quad I, J, \cdots = 1, 2\,, \quad \Rightarrow A^{IJ}{}_\mu = A^0{}_\mu \varepsilon^{IJ}\,.$$

The (n) $N = 2$ vector multiplets are

$$\{A^i{}_\mu, \lambda^i{}_I, Z^i\}\,, \quad i = 1, \cdots, n\,, \quad \Rightarrow A^\Lambda{}_\mu\,, \quad \Lambda = 0, \cdots, n\,.$$

The (m) hypermultiplets are

$$\{\zeta_\alpha, q^u\}\,, \quad u = 1, \cdots, 4m\,, \quad \alpha = 1, \cdots, 2m\,.$$

The n complex scalars are encoded into the $2\bar{n}$-dimensional symplectic section $(\bar{n} = 1 + n)$

$$\mathcal{V} = \begin{pmatrix} \mathcal{L}^\Lambda \\ \mathcal{M}_\Lambda \end{pmatrix}\,, \qquad \langle \mathcal{V} \mid \mathcal{V}^* \rangle = -2i\,.$$

2 – Review of the N=2 case

Since the timelike supersymmetric solutions of $N > 2$ turn out to be related to those of $N = 2$ theories (Hübscher, Meessen & O. (2006)), we briefly review them first.

The $N = 2$ supergravity multiplet is

$$\{e^{a}{}_{\mu}, \psi_{I\,\mu}, A^{IJ}{}_{\mu}\}\,, \quad I, J, \cdots = 1, 2\,, \quad \Rightarrow A^{IJ}{}_{\mu} = A^{0}{}_{\mu}\varepsilon^{IJ}\,.$$

The (n) $N = 2$ vector multiplets are

$$\{A^{i}{}_{\mu}, \lambda^{i}{}_{I}, Z^{i}\}\,, \quad i = 1, \cdots, n\,, \quad \Rightarrow A^{\Lambda}{}_{\mu}\,, \quad \Lambda = 0, \cdots, n\,.$$

The (m) hypermultiplets are

$$\{\zeta_{\alpha}, q^{u}\}\,, \quad u = 1, \cdots, 4m\,, \quad \alpha = 1, \cdots, 2m\,.$$

The n complex scalars are encoded into the $2\bar{n}$-dimensional symplectic section $(\bar{n} = 1 + n)$

$$\mathcal{V} = \begin{pmatrix} \mathcal{L}^{\Lambda} \\ \mathcal{M}_{\Lambda} \end{pmatrix}, \qquad \langle \mathcal{V} \mid \mathcal{V}^{*} \rangle = -2i\,.$$

This is a extremely redundant (but useful) description of the scalars .

The supersymmetry transformations of the fermions are

$$\begin{aligned}
\delta_\epsilon \psi_{I\,\mu} &= \mathfrak{D}_\mu \epsilon_I + \varepsilon_{IJ}\, T^+{}_{\mu\nu}\gamma^\nu\, \epsilon^J\,, \\
\delta_\epsilon \lambda^{iI} &= i\, \not{\partial} Z^i \epsilon^I \;+\; \varepsilon^{IJ}\, \not{G}^{i\,+}\, \epsilon_J\,. \\
\delta_\epsilon \zeta_\alpha &= -i \mathbb{C}_{\alpha\beta}\, \mathsf{U}^{\beta I}{}_u\, \varepsilon_{IJ}\;\; \not{\partial} q^u\, \epsilon^J\,,
\end{aligned}$$

The supersymmetry transformations of the fermions are

$$\delta_\epsilon \psi_{I\,\mu} = \mathfrak{D}_\mu \epsilon_I + \varepsilon_{IJ}\, T^+{}_{\mu\nu}\gamma^\nu\, \epsilon^J\,,$$

$$\delta_\epsilon \lambda^{iI} = i\, \not\partial Z^i \epsilon^I \,+\, \varepsilon^{IJ}\, \not G^{i\,+}\, \epsilon_J\,.$$

$$\delta_\epsilon \zeta_\alpha = -i\mathbb{C}_{\alpha\beta}\, \mathsf{U}^{\beta I}{}_u\, \varepsilon_{IJ}\, \not\partial q^u\, \epsilon^J\,,$$

where the graviphoton and matter vector field strengths are

$$T^+ = \langle\, \mathcal{V}\mid \mathcal{F}^+\,\rangle\,,\qquad G^{i\,+} = \tfrac{i}{2}\mathcal{G}^{ij^*}\langle\, \mathcal{D}_{j^*}\mathcal{V}^*\mid \mathcal{F}^+\,\rangle\,,\qquad \mathcal{F}^+ \equiv \begin{pmatrix} F^{\Lambda\,+} \\ \mathcal{N}^*_{\Lambda\Sigma}F^{\Sigma\,+} \end{pmatrix}\,,$$

The supersymmetry transformations of the fermions are

$$\delta_\epsilon \psi_{I\,\mu} = \mathfrak{D}_\mu \epsilon_I + \varepsilon_{IJ}\, T^+{}_{\mu\nu}\gamma^\nu\ \epsilon^J\,,$$

$$\delta_\epsilon \lambda^{iI} = i\ \not\partial Z^i \epsilon^I\ +\ \varepsilon^{IJ}\ \not G^{i\,+}\ \epsilon_J\,.$$

$$\delta_\epsilon \zeta_\alpha = -i\mathbb{C}_{\alpha\beta}\ \mathsf{U}^{\beta I}{}_u\ \varepsilon_{IJ}\ \ \not\partial q^u\ \epsilon^J\,,$$

where the graviphoton and matter vector field strengths are

$$T^+ = \langle\, \mathcal{V} \mid \mathcal{F}^+ \,\rangle\,, \qquad G^{i\,+} = \tfrac{i}{2}\mathcal{G}^{ij^*}\langle\, \mathcal{D}_{j^*}\mathcal{V}^* \mid \mathcal{F}^+ \,\rangle\,, \qquad \mathcal{F}^+ \equiv \begin{pmatrix} F^{\Lambda\,+} \\ \mathcal{N}^*_{\Lambda\Sigma}F^{\Sigma\,+} \end{pmatrix},$$

$\mathfrak{D}$ is the Lorentz-, Kähler- and $SU(2)$- covariant derivative (Kähler $+\ SU(2) = U(2)$)

$$\mathfrak{D}_\mu \epsilon_I = (\partial_\mu + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\ +\ \tfrac{i}{2}\ \mathcal{Q}_\mu)\ \epsilon_I\ +\ \mathsf{A}_{\mu\,I}{}^J\ \epsilon_J\,,$$

and where $\mathsf{U}^{\alpha I}{}_u(q)$ is the *Quadbein*.

The supersymmetry transformations of the fermions are

$$\begin{aligned} \delta_\epsilon \psi_{I\,\mu} &= \mathfrak{D}_\mu \epsilon_I + \varepsilon_{IJ}\, T^+{}_{\mu\nu}\gamma^\nu\ \epsilon^J\,, \\ \delta_\epsilon \lambda^{iI} &= i\ \not\partial Z^i \epsilon^I\ +\ \varepsilon^{IJ}\ \not G^{i\,+}\ \epsilon_J\,. \\ \delta_\epsilon \zeta_\alpha &= -i\mathbb{C}_{\alpha\beta}\ \mathsf{U}^{\beta I}{}_u\ \varepsilon_{IJ}\ \ \not\partial q^u\ \epsilon^J\,, \end{aligned}$$

where the graviphoton and matter vector field strengths are

$$T^+ = \langle\, \mathcal{V} \mid \mathcal{F}^+ \,\rangle\,, \qquad G^{i\,+} = \tfrac{i}{2}\mathcal{G}^{ij^*}\langle\, \mathcal{D}_{j^*}\mathcal{V}^* \mid \mathcal{F}^+ \,\rangle\,, \qquad \mathcal{F}^+ \equiv \begin{pmatrix} F^{\Lambda\,+} \\ \mathcal{N}^*_{\Lambda\Sigma}F^{\Sigma\,+} \end{pmatrix}\,,$$

$\mathfrak{D}$ is the Lorentz-, Kähler- and $SU(2)$- covariant derivative (Kähler $+\ SU(2) = U(2)$)

$$\mathfrak{D}_\mu \epsilon_I = (\partial_\mu + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\ +\ \tfrac{i}{2}\ \mathcal{Q}_\mu)\ \epsilon_I\ +\ \mathsf{A}_{\mu\,I}{}^J\ \epsilon_J\,,$$

and where $\mathsf{U}^{\alpha I}{}_u(q)$ is the *Quadbein*. The action for the bosonic fields is

$$\begin{aligned} S \ = \ \int d^4x\sqrt{|g|}\ \big[& R + 2\mathcal{G}_{ij^*}\partial_\mu Z^i \partial^\mu Z^{*j^*} + 2\mathsf{H}_{uv}\partial_\mu q^u \partial^\mu q^v \\ & +2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}F^\Sigma{}_{\mu\nu} - 2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}\star F^\Sigma{}_{\mu\nu}\big]\,. \end{aligned}$$

3 – The $N = 2$ Killing Spinor Equations (KSEs)

3 – The $N = 2$ Killing Spinor Equations (KSEs)

They take the form

$$
\begin{aligned}
\mathfrak{D}_\mu \epsilon_I + \varepsilon_{IJ}\, T^+{}_{\mu\nu}\gamma^\nu\, \epsilon^J &= 0\,,\\
i\, \not{\partial} Z^i \epsilon^I + \varepsilon^{IJ}\, \not{G}^{i\,+}\, \epsilon_J &= 0\,,\\
-i\mathbb{C}_{\alpha\beta}\, \mathsf{U}^{\beta I}{}_u\, \varepsilon_{IJ}\, \not{\partial} q^u\, \epsilon^J &= 0\,.
\end{aligned}
$$

3 – The $N = 2$ Killing Spinor Equations (KSEs)

They take the form

$$\begin{aligned} \mathfrak{D}_\mu \epsilon_I + \varepsilon_{IJ}\, T^+{}_{\mu\nu}\gamma^\nu\, \epsilon^J &= 0\,, \\ i\, \not\partial Z^i \epsilon^I \;+\; \varepsilon^{IJ}\, \not G^{i\,+}\, \epsilon_J &= 0\,, \\ -i\mathbb{C}_{\alpha\beta}\, \mathsf{U}^{\beta I}{}_u\, \varepsilon_{IJ}\;\; \not\partial q^u\, \epsilon^J &= 0\,. \end{aligned}$$

The goal is to find **all** the bosonic field configurations $\{e^a{}_\mu, A^\Lambda{}_\mu, Z^i, q^u\}$ such that the above KSEs admit at least one solution ϵ^I.

The **spinor-bilinear method** consists in the following steps:

The **spinor-bilinear method** consists in the following steps:

1. Assume that one has a bosonic field configuration such that ϵ^I exists.

The **spinor-bilinear method** consists in the following steps:

1. Assume that one has a bosonic field configuration such that ϵ^I exists.

2. Construct all the independent bilinears with the **commuting** Killing spinor ϵ^I and find the equations they satisfy:

The **spinor-bilinear method** consists in the following steps:

1. Assume that one has a bosonic field configuration such that ϵ^I exists.

2. Construct all the independent bilinears with the **commuting** Killing spinor ϵ^I and find the equations they satisfy:

   (a) Due to the Fierz identities. (*Spinor-bilinear algebra*)

The **spinor-bilinear method** consists in the following steps:

1. Assume that one has a bosonic field configuration such that ϵ^I exists.

2. Construct all the independent bilinears with the **commuting** Killing spinor ϵ^I and find the equations they satisfy:

   (a) Due to the Fierz identities. (*Spinor-bilinear algebra*)

   (b) Due to the KSEs.

The **spinor-bilinear method** consists in the following steps:

1. Assume that one has a bosonic field configuration such that ϵ^I exists.

2. Construct all the independent bilinears with the **commuting** Killing spinor ϵ^I and find the equations they satisfy:

   (a) Due to the Fierz identities. (*Spinor-bilinear algebra*)

   (b) Due to the KSEs.

3. Find their integrability conditions and show that they are also sufficient to solve the KSEs. At this point all supersymmetric configurations are determined.

The **spinor-bilinear method** consists in the following steps:

1. Assume that one has a bosonic field configuration such that ϵ^I exists.
2. Construct all the independent bilinears with the **commuting** Killing spinor ϵ^I and find the equations they satisfy:
   (a) Due to the Fierz identities. (*Spinor-bilinear algebra*)
   (b) Due to the KSEs.
3. Find their integrability conditions and show that they are also sufficient to solve the KSEs. At this point all supersymmetric configurations are determined.
4. Determine which equations of motion are independent for supersymmetric configurations. This is determined by the Killing Spinor Identities (KSIs).

The **spinor-bilinear method** consists in the following steps:

1. Assume that one has a bosonic field configuration such that ϵ^I exists.
2. Construct all the independent bilinears with the **commuting** Killing spinor ϵ^I and find the equations they satisfy:
   (a) Due to the Fierz identities. (*Spinor-bilinear algebra*)
   (b) Due to the KSEs.
3. Find their integrability conditions and show that they are also sufficient to solve the KSEs. At this point all supersymmetric configurations are determined.
4. Determine which equations of motion are independent for supersymmetric configurations. This is determined by the Killing Spinor Identities (KSIs).
5. Impose the independent equations of motion on the supersymmetric configurations we just identified.

4 − The $N = 2$ spinor-bilinears algebra

The independent bilinears that we can construct with one $U(2)$ vector of Weyl spinors ϵ_I are:

4 − The $N = 2$ spinor-bilinears algebra

The independent bilinears that we can construct with one $U(2)$ vector of Weyl spinors ϵ_I are:

1. A complex antisymmetric matrix of scalars $M_{IJ} \equiv \bar{\epsilon}_I \epsilon_J = X \varepsilon_{IJ}$.
   X is an $SU(2)$ singlet but has $U(1)$ Kähler weight.

4 − The $N = 2$ spinor-bilinears algebra

The independent bilinears that we can construct with one $U(2)$ vector of Weyl spinors ϵ_I are:

1. A complex antisymmetric matrix of scalars $M_{IJ} \equiv \bar{\epsilon}_I \epsilon_J = X \varepsilon_{IJ}$.
   X is an $SU(2)$ singlet but has $U(1)$ Kähler weight.

2. A Hermitean matrix of vectors $V^I{}_{J\,a} \equiv i\bar{\epsilon}^I \gamma_a \epsilon_J$.

4 − The $N = 2$ spinor-bilinears algebra

The independent bilinears that we can construct with one $U(2)$ vector of Weyl spinors ϵ_I are:

1. A complex antisymmetric matrix of scalars $M_{IJ} \equiv \bar{\epsilon}_I \epsilon_J = X \varepsilon_{IJ}$.
   X is an $SU(2)$ singlet but has $U(1)$ Kähler weight.

2. A Hermitean matrix of vectors $V^I{}_{J\,a} \equiv i\bar{\epsilon}^I \gamma_a \epsilon_J$.

The 4-d Fierz identities imply that $V_a \equiv V^I{}_{I\,a}$ is always non-spacelike:

$$V^2 = -V^I{}_J \cdot V^J{}_I = 2M^{IJ} M_{IJ} = 4|X|^2 \geq 0\,.$$

We only consider the timelike case $X \neq 0$ in which all $V^I{}_{J\,a}$ are independent.

4 − The $N = 2$ spinor-bilinears algebra

The independent bilinears that we can construct with one $U(2)$ vector of Weyl spinors ϵ_I are:

1. A complex antisymmetric matrix of scalars $M_{IJ} \equiv \bar{\epsilon}_I \epsilon_J = X \varepsilon_{IJ}$.
   X is an $SU(2)$ singlet but has $U(1)$ Kähler weight.

2. A Hermitean matrix of vectors $V^I{}_{J\,a} \equiv i\bar{\epsilon}^I \gamma_a \epsilon_J$.

The 4-d Fierz identities imply that $V_a \equiv V^I{}_{I\,a}$ is always non-spacelike:

$$V^2 = -V^I{}_J \cdot V^J{}_I = 2M^{IJ} M_{IJ} = 4|X|^2 \geq 0\,.$$

We only consider the timelike case $X \neq 0$ in which all $V^I{}_{J\,a}$ are independent.

With them one can construct a tetrad

$$V^a{}_\mu \equiv \frac{1}{\sqrt{2}} V^I{}_{J\,\mu} (\sigma^a)^J{}_I\,, \qquad V^I{}_{J\,\mu} = \frac{1}{\sqrt{2}} V^a{}_\mu (\sigma^a)^I{}_J\,,$$

with $\sigma^0 = 1$ and σ^m the 2×2 Pauli matrices as an orthonormal tetrad in which $V^0 = \sqrt{2} V$ is timelike and the V^ms are spacelike.

4 − The $N = 2$ spinor-bilinears algebra

The independent bilinears that we can construct with one $U(2)$ vector of Weyl spinors ϵ_I are:

1. A complex antisymmetric matrix of scalars $M_{IJ} \equiv \bar{\epsilon}_I \epsilon_J = X \varepsilon_{IJ}$. X is an $SU(2)$ singlet but has $U(1)$ Kähler weight.

2. A Hermitean matrix of vectors $V^I{}_{J\,a} \equiv i\bar{\epsilon}^I \gamma_a \epsilon_J$.

The 4-d Fierz identities imply that $V_a \equiv V^I{}_{I\,a}$ is always non-spacelike:

$$V^2 = -V^I{}_J \cdot V^J{}_I = 2M^{IJ} M_{IJ} = 4|X|^2 \geq 0\,.$$

We only consider the timelike case $X \neq 0$ in which all $V^I{}_{J\,a}$ are independent.

With them one can construct a tetrad

$$V^a{}_\mu \equiv \tfrac{1}{\sqrt{2}} V^I{}_{J\,\mu} (\sigma^a)^J{}_I\,, \qquad\qquad V^I{}_{J\,\mu} = \tfrac{1}{\sqrt{2}} V^a{}_\mu (\sigma^a)^I{}_J\,,$$

with $\sigma^0 = 1$ and σ^m the 2×2 Pauli matrices as an orthonormal tetrad in which $V^0 = \sqrt{2} V$ is timelike and the V^ms are spacelike. (**This will not work for $N > 2$!**)

5 – The $N = 2$ Killing Spinor Identities (KSI)s

If we assume that a given bosonic field configuration admits a Killing spinor ϵ_I, then we find that the (*off-shell*) "equations of motion" $\{\mathcal{E}^{\mu\nu}, \mathcal{E}^{\mu}, \mathcal{E}^{i}, \mathcal{E}_{u}\}$ satisfy the KSIs:

5 – The $N = 2$ Killing Spinor Identities (KSI)s

If we assume that a given bosonic field configuration admits a Killing spinor ϵ_I, then we find that the (*off-shell*) "equations of motion" $\{\mathcal{E}^{\mu\nu}, \mathcal{E}^{\mu}, \mathcal{E}^{i}, \mathcal{E}_{u}\}$ satisfy the KSIs:

1. $\mathcal{E}^{0m} = \mathcal{E}^{mn} = 0.$

5 – The $N = 2$ Killing Spinor Identities (KSI)s

If we assume that a given bosonic field configuration admits a Killing spinor ϵ_I, then we find that the (*off-shell*) "equations of motion" $\{\mathcal{E}^{\mu\nu}, \mathcal{E}^{\mu}, \mathcal{E}^{i}, \mathcal{E}_{u}\}$ satisfy the KSIs:

1. $\mathcal{E}^{0m} = \mathcal{E}^{mn} = 0.$

2. $\mathcal{E}^{m} = 0.$

5 – The $N = 2$ Killing Spinor Identities (KSI)s

If we assume that a given bosonic field configuration admits a Killing spinor ϵ_I, then we find that the (*off-shell*) "equations of motion" $\{\mathcal{E}^{\mu\nu}, \mathcal{E}^{\mu}, \mathcal{E}^{i}, \mathcal{E}_{u}\}$ satisfy the KSIs:

1. $\mathcal{E}^{0m} = \mathcal{E}^{mn} = 0.$

2. $\mathcal{E}^{m} = 0.$

3. $\mathcal{E}_{u} = 0$, ($\Rightarrow$ no attractor mechanism for hyperscalars)

5 – The $N = 2$ Killing Spinor Identities (KSI)s

If we assume that a given bosonic field configuration admits a Killing spinor ϵ_I, then we find that the (*off-shell*) "equations of motion" $\{\mathcal{E}^{\mu\nu}, \mathcal{E}^{\mu}, \mathcal{E}^{i}, \mathcal{E}_{u}\}$ satisfy the KSIs:

1. $\mathcal{E}^{0m} = \mathcal{E}^{mn} = 0.$

2. $\mathcal{E}^{m} = 0.$

3. $\mathcal{E}_{u} = 0$, ($\Rightarrow$ no attractor mechanism for hyperscalars)

4. $\mathcal{E}^{00} = -4|X|\langle\, \mathcal{E}^{0} \mid \Re\mathrm{e}(\mathcal{V}/X)\,\rangle$, (Bogomol'nyi bound)

5 – The $N=2$ Killing Spinor Identities (KSI)s

If we assume that a given bosonic field configuration admits a Killing spinor ϵ_I, then we find that the (*off-shell*) "equations of motion" $\{\mathcal{E}^{\mu\nu}, \mathcal{E}^{\mu}, \mathcal{E}^{i}, \mathcal{E}_{u}\}$ satisfy the KSIs:

1. $\mathcal{E}^{0m} = \mathcal{E}^{mn} = 0$.

2. $\mathcal{E}^{m} = 0$.

3. $\mathcal{E}_{u} = 0$, ($\Rightarrow$ no attractor mechanism for hyperscalars)

4. $\mathcal{E}^{00} = -4|X|\langle\, \mathcal{E}^{0} \mid \Re\mathrm{e}(\mathcal{V}/X)\,\rangle$, (Bogomol'nyi bound)

5. $0 = \langle\, \mathcal{E}^{0} \mid \Im\mathrm{m}(\mathcal{V}/X)\,\rangle$, ($\Rightarrow$ no NUT charges) (Bellorín, Meessen, Ortín (2008)).

5 – The $N=2$ Killing Spinor Identities (KSI)s

If we assume that a given bosonic field configuration admits a Killing spinor ϵ_I, then we find that the (*off-shell*) "equations of motion" $\{\mathcal{E}^{\mu\nu}, \mathcal{E}^{\mu}, \mathcal{E}^{i}, \mathcal{E}_{u}\}$ satisfy the KSIs:

1. $\mathcal{E}^{0m} = \mathcal{E}^{mn} = 0$.

2. $\mathcal{E}^{m} = 0$.

3. $\mathcal{E}_{u} = 0$, ($\Rightarrow$ no attractor mechanism for hyperscalars)

4. $\mathcal{E}^{00} = -4|X|\langle\, \mathcal{E}^{0} \mid \Re\mathrm{e}(\mathcal{V}/X)\,\rangle$, (Bogomol'nyi bound)

5. $0 = \langle\, \mathcal{E}^{0} \mid \Im\mathrm{m}(\mathcal{V}/X)\,\rangle$, ($\Rightarrow$ no NUT charges) (Bellorín, Meessen, Ortín (2008)).

6. $\mathcal{E}_{i^*} = 2\left(\frac{X}{X^*}\right)^{1/2}\langle\, \mathcal{E}^{0} \mid \mathcal{D}_{i^*}\mathcal{V}^*\,\rangle$, ($\Rightarrow$ attractor mechanism)

The only independent equations of motion that have to be imposed on $N = 2$, $d = 4$ supersymmetric configurations are

$$\mathcal{E}^0 = 0\,.$$

6 – The $N = 2$ supersymmetric solutions

They can be constructed as follows:

6 – The $N = 2$ supersymmetric solutions

They can be constructed as follows:

1. Define the $U(1)$-neutral real symplectic vectors $\mathcal{R}$ and $\mathcal{I}$

$$\mathcal{R} + i\mathcal{I} \equiv \mathcal{V}/X\,.$$

($\Rightarrow$ No Kähler nor $SU(2)$ gauge -fixing is necessary!)

6 – The $N = 2$ supersymmetric solutions

They can be constructed as follows:

1. Define the $U(1)$-neutral real symplectic vectors $\mathcal{R}$ and $\mathcal{I}$

$$\mathcal{R} + i\mathcal{I} \equiv \mathcal{V}/X \, .$$

($\Rightarrow$ No Kähler nor $SU(2)$ gauge -fixing is necessary!)

2. The components of $\mathcal{I}$ are given by a symplectic vector real functions $\mathcal{H}$ harmonic in the 3-dimensional transverse space with metric $\gamma_{\underline{mn}}$:

6 – The $N = 2$ supersymmetric solutions

They can be constructed as follows:

1. Define the $U(1)$-neutral real symplectic vectors $\mathcal{R}$ and $\mathcal{I}$

$$\mathcal{R} + i\mathcal{I} \equiv \mathcal{V}/X \, .$$

($\Rightarrow$ No Kähler nor $SU(2)$ gauge -fixing is necessary!)

2. The components of $\mathcal{I}$ are given by a symplectic vector real functions $\mathcal{H}$ harmonic in the 3-dimensional transverse space with metric $\gamma_{\underline{mn}}$:

$$\boxed{\nabla^2_{(3)}\mathcal{H} = 0.}$$

6 – The $N = 2$ supersymmetric solutions

They can be constructed as follows:

1. Define the $U(1)$-neutral real symplectic vectors $\mathcal{R}$ and $\mathcal{I}$

$$\mathcal{R} + i\mathcal{I} \equiv \mathcal{V}/X\,.$$

($\Rightarrow$ No Kähler nor $SU(2)$ gauge -fixing is necessary!)

2. The components of $\mathcal{I}$ are given by a symplectic vector real functions $\mathcal{H}$ harmonic in the 3-dimensional transverse space with metric $\gamma_{\underline{mn}}$:

$$\boxed{\nabla^2_{(3)}\mathcal{H} = 0.}$$

3. $\mathcal{R}$ is to be found from $\mathcal{I}$ by solving the generalized *stabilization equations* (using the redundancy of $\mathcal{V}$).

6 – The $N = 2$ supersymmetric solutions

They can be constructed as follows:

1. Define the $U(1)$-neutral real symplectic vectors $\mathcal{R}$ and $\mathcal{I}$

$$\mathcal{R} + i\mathcal{I} \equiv \mathcal{V}/X\,.$$

($\Rightarrow$ No Kähler nor $SU(2)$ gauge -fixing is necessary!)

2. The components of $\mathcal{I}$ are given by a symplectic vector real functions $\mathcal{H}$ harmonic in the 3-dimensional transverse space with metric $\gamma_{\underline{mn}}$:

$$\boxed{\nabla^2_{(3)}\mathcal{H} = 0.}$$

3. $\mathcal{R}$ is to be found from $\mathcal{I}$ by solving the generalized *stabilization equations* (using the redundancy of $\mathcal{V}$).

4. The scalars Z^i are given by the quotients

$$Z^i = \frac{\mathcal{V}^i/X}{\mathcal{V}^0/X} = \frac{\mathcal{R}^i + i\mathcal{I}^i}{\mathcal{R}^0 + i\mathcal{I}^0}\,.$$

5. The hyperscalars $q^{u}(x)$ are the mappings satisfying

$$\mathsf{U}^{\alpha J}{}_{m}\,(\sigma^{m})_{J}{}^{I} = 0\,, \qquad \mathsf{U}^{\alpha J}{}_{n} \equiv V_{n}{}^{\underline{m}}\partial_{\underline{m}}q^{u}\,\mathsf{U}^{\alpha J}{}_{u}\,.$$

5. The hyperscalars $q^u(x)$ are the mappings satisfying

$$\mathsf{U}^{\alpha J}{}_{m}\,(\sigma^m)_J{}^I \;=\; 0\,, \qquad\qquad \mathsf{U}^{\alpha J}{}_{n} \;\equiv\; V_n{}^{\underline{m}}\partial_{\underline{m}} q^u\; \mathsf{U}^{\alpha J}{}_{u}\,.$$

6. The metric takes the form

$$ds^2 \;=\; 2|X|^2(dt+\omega)^2 - \frac{1}{2|X|^2}\gamma_{\underline{mn}}dx^m dx^n\,.$$

5. The hyperscalars $q^u(x)$ are the mappings satisfying

$$\mathsf{U}^{\alpha J}{}_m \, (\sigma^m)_J{}^I \;=\; 0\,, \qquad\qquad \mathsf{U}^{\alpha J}{}_n \;\equiv\; V_n{}^{\underline{m}}\partial_{\underline{m}}q^u \; \mathsf{U}^{\alpha J}{}_u\,.$$

6. The metric takes the form

$$ds^2 \;=\; 2|X|^2(dt+\omega)^2 - \frac{1}{2|X|^2}\gamma_{\underline{mn}}dx^m dx^n\,.$$

where

$$\frac{1}{2|X|^2} = \langle\,\mathcal{R}\mid\mathcal{I}\,\rangle\,, \qquad\qquad (d\omega)_{mn} = 2\epsilon_{mnp}\langle\,\mathcal{I}\mid\partial^p\mathcal{I}\,\rangle\,.$$

5. The hyperscalars $q^u(x)$ are the mappings satisfying

$$\mathsf{U}^{\alpha J}{}_{m}\,(\sigma^m)_J{}^I \;=\; 0\,, \qquad\qquad \mathsf{U}^{\alpha J}{}_{n} \;\equiv\; V_n{}^{\underline{m}}\partial_{\underline{m}}q^u\;\mathsf{U}^{\alpha J}{}_{u}\,.$$

6. The metric takes the form

$$ds^2 \;=\; 2|X|^2(dt+\omega)^2 - \frac{1}{2|X|^2}\gamma_{\underline{mn}}dx^m dx^n\,.$$

where

$$\frac{1}{2|X|^2} = \langle\,\mathcal{R}\mid\mathcal{I}\,\rangle\,, \qquad\qquad (d\omega)_{mn} = 2\epsilon_{mnp}\langle\,\mathcal{I}\mid\partial^p\mathcal{I}\,\rangle\,.$$

$\gamma_{\underline{mn}}$ is determined indirectly from the hyperscalars : its spin connection ϖ^{mn} in the basis $\{V^m\}$ is related to the pullback of the $SU(2)$ connection of the hyper-Kähler manifold $\mathsf{A}^I{}_{J\mu} = \frac{1}{\sqrt{2}}\mathsf{A}^m{}_u(\sigma^m)^I{}_J\partial_\mu q^u$, by

$$\varpi_m{}^{np} = \varepsilon^{npq}\mathsf{A}^q{}_m\,.$$

5. The hyperscalars $q^u(x)$ are the mappings satisfying

$$\mathsf{U}^{\alpha J}{}_m\,(\sigma^m)_J{}^I \;=\; 0\,, \qquad\qquad \mathsf{U}^{\alpha J}{}_n \;\equiv\; V_n{}^{\underline{m}}\partial_{\underline{m}} q^u\; \mathsf{U}^{\alpha J}{}_u\,.$$

6. The metric takes the form

$$ds^2 \;=\; 2|X|^2(dt+\omega)^2 - \frac{1}{2|X|^2}\gamma_{\underline{mn}}dx^m dx^n\,.$$

where

$$\frac{1}{2|X|^2} = \langle\,\mathcal{R}\mid\mathcal{I}\,\rangle\,, \qquad\qquad (d\omega)_{mn} = 2\epsilon_{mnp}\langle\,\mathcal{I}\mid\partial^p\mathcal{I}\,\rangle\,.$$

$\gamma_{\underline{mn}}$ is determined indirectly from the hyperscalars : its spin connection ϖ^{mn} in the basis $\{V^m\}$ is related to the pullback of the $SU(2)$ connection of the hyper-Kähler manifold $\mathsf{A}^I{}_{J\mu} = \frac{1}{\sqrt{2}}\mathsf{A}^m{}_u(\sigma^m)^I{}_J\partial_\mu q^u$, by

$$\varpi_m{}^{np} = \varepsilon^{npq}\mathsf{A}^q{}_m\,.$$

7. The vector field strengths are

$$\mathcal{F} = -\tfrac{1}{2}d(\mathcal{R}\hat{V}) - \tfrac{1}{2}\star(\hat{V}\wedge d\mathcal{I})\,, \qquad\qquad \hat{V} = 2\sqrt{2}|X|^2(dt+\omega)\,.$$

7 – The all-N formulation of 4-d sugras

7 – The all-N formulation of 4-d sugras

All 4-d supergravity multiplets can be written in the form

$$\left\{e^{a}{}_{\mu}, \psi_{I\,\mu}, A^{IJ}{}_{\mu}, \chi_{IJK}, P_{IJKL\,\mu}, \chi^{IJKLM}\right\}\,, \quad I, J, \cdots = 1, \cdots, N\,,$$

7 – The all-N formulation of 4-d sugras

All 4-d supergravity multiplets can be written in the form

$$\left\{e^{a}{}_{\mu}, \psi_{I\,\mu}, A^{IJ}{}_{\mu}, \chi_{IJK}, P_{IJKL\,\mu}, \chi^{IJKLM}\right\}, \quad I, J, \cdots = 1, \cdots, N,$$

All vector multiplets can be written in the form

$$\left\{A_{i\,\mu}, \lambda_{iI}, P_{iIJ\,\mu}, \lambda_{i}{}^{IJK}\right\}, \quad i = 1, \cdots, n.$$

7 – The all-N formulation of 4-d sugras

All 4-d supergravity multiplets can be written in the form

$$\left\{e^a{}_\mu, \psi_{I\,\mu}, A^{IJ}{}_\mu, \chi_{IJK}, P_{IJKL\,\mu}, \chi^{IJKLM}\right\}, \quad I, J, \cdots = 1, \cdots, N,$$

All vector multiplets can be written in the form

$$\left\{A_{i\,\mu}, \lambda_{iI}, P_{iIJ\,\mu}, \lambda_i{}^{IJK}\right\}, \quad i = 1, \cdots, n.$$

The price to pay for using this representation is that all the fields that can be related by $SU(N)$ duality relations, are:

- $N = 4$: $P^{*\,i\,IJ} = \frac{1}{2}\varepsilon^{IJKL}P_{i\,KL}$, and $\lambda_{iI} = \frac{1}{3!}\varepsilon_{IJKL}\lambda_i{}^{IJK}$.
- $N = 6$: $P^{*\,IJ} = \frac{1}{4!}\varepsilon^{IJK_1\cdots K_4}P_{K_1\cdots K_4}$, $\chi_{IJK} = \frac{1}{3!}\varepsilon_{IJKLMN}\lambda^{IJK}$, and $\chi^{I_1\cdots I_5} = \varepsilon^{I_1\cdots I_5 J}\lambda_J$.
- $N = 8$: $P^{*\,I_1\cdots I_4} = \frac{1}{4!}\varepsilon^{I_1\cdots I_4 J_1\cdots J_4}P_{J_1\cdots J_4}$, and $\chi_{I_1 I_2 I_3} = \frac{1}{5!}\varepsilon_{I_1 I_2 I_3 J_1\cdots J_5}\chi^{J_1\cdots J_5}$.
  These constraints must be taken into account in the action.

The scalars are encoded into the $2\bar{n}$-dimensional ($\bar{n} \equiv n + \frac{N(N-1)}{2}$) symplectic vectors

$$\mathcal{V}_{IJ} = \left(\begin{array}{c} f^{\Lambda}{}_{IJ} \\ h_{\Lambda\, IJ} \end{array} \right) , \qquad \text{and} \qquad \mathcal{V}_{i} = \left(\begin{array}{c} f^{\Lambda}{}_{i} \\ h_{\Lambda\, i} \end{array} \right) , \qquad \Lambda = 1, \cdots , \bar{n} ,$$

normalized

$$\langle \mathcal{V}_{IJ} \mid \mathcal{V}^{*\, KL} \rangle = -2i\delta^{KL}{}_{IJ} , \qquad \langle \mathcal{V}_{i} \mid \mathcal{V}^{*\, j} \rangle = -i\delta_{i}{}^{j} .$$

The scalars are encoded into the $2\bar{n}$-dimensional ($\bar{n} \equiv n + \frac{N(N-1)}{2}$) symplectic vectors

$$\mathcal{V}_{IJ} = \begin{pmatrix} f^{\Lambda}{}_{IJ} \\ h_{\Lambda\, IJ} \end{pmatrix}, \qquad \text{and} \qquad \mathcal{V}_{i} = \begin{pmatrix} f^{\Lambda}{}_{i} \\ h_{\Lambda\, i} \end{pmatrix}, \qquad \Lambda = 1, \cdots, \bar{n}\,,$$

normalized

$$\langle \mathcal{V}_{IJ} \mid \mathcal{V}^{*\, KL} \rangle = -2i\delta^{KL}{}_{IJ}\,, \qquad \langle \mathcal{V}_{i} \mid \mathcal{V}^{*\, j} \rangle = -i\delta_{i}{}^{j}\,.$$

They can be combined into the $Usp(\bar{n}, \bar{n})$ matrix

$$U \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} f + ih & f^{*} + ih^{*} \\ f - ih & f^{*} - ih^{*} \end{pmatrix}.$$

The scalars are encoded into the $2\bar{n}$-dimensional $(\bar{n} \equiv n + \frac{N(N-1)}{2})$ symplectic vectors

$$\mathcal{V}_{IJ} = \begin{pmatrix} f^{\Lambda}{}_{IJ} \\ h_{\Lambda\, IJ} \end{pmatrix}, \qquad \text{and} \qquad \mathcal{V}_{i} = \begin{pmatrix} f^{\Lambda}{}_{i} \\ h_{\Lambda\, i} \end{pmatrix}, \qquad \Lambda = 1, \cdots, \bar{n}\,,$$

normalized

$$\langle \mathcal{V}_{IJ} \mid \mathcal{V}^{*\,KL} \rangle = -2i\delta^{KL}{}_{IJ}\,, \qquad \langle \mathcal{V}_{i} \mid \mathcal{V}^{*\,j} \rangle = -i\delta_{i}{}^{j}\,.$$

They can be combined into the $Usp(\bar{n}, \bar{n})$ matrix

$$U \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} f + ih & f^{*} + ih^{*} \\ f - ih & f^{*} - ih^{*} \end{pmatrix}.$$

They generalize the $N = 2$ sections

$$\mathcal{V}_{IJ} = \mathcal{V}\varepsilon_{IJ}\,, = \begin{pmatrix} \mathcal{L}^{\Lambda}\varepsilon_{IJ} \\ \mathcal{M}_{\Lambda}\varepsilon_{IJ} \end{pmatrix}, \qquad \text{and} \qquad \mathcal{V}_{i} = \mathcal{D}_{i}\mathcal{V} = \begin{pmatrix} f^{\Lambda}{}_{i} \\ h_{\Lambda\, i} \end{pmatrix}.$$

The scalars are encoded into the $2\bar{n}$-dimensional ($\bar{n} \equiv n + \frac{N(N-1)}{2}$) symplectic vectors

$$\mathcal{V}_{IJ} = \begin{pmatrix} f^{\Lambda}{}_{IJ} \\ h_{\Lambda\, IJ} \end{pmatrix}, \qquad \text{and} \qquad \mathcal{V}_{i} = \begin{pmatrix} f^{\Lambda}{}_{i} \\ h_{\Lambda\, i} \end{pmatrix}, \qquad \Lambda = 1, \cdots, \bar{n}\,,$$

normalized

$$\langle \mathcal{V}_{IJ} \mid \mathcal{V}^{*\,KL} \rangle = -2i\delta^{KL}{}_{IJ}\,, \qquad \langle \mathcal{V}_{i} \mid \mathcal{V}^{*\,j} \rangle = -i\delta_{i}{}^{j}\,.$$

They can be combined into the $Usp(\bar{n}, \bar{n})$ matrix

$$U \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} f + ih & f^* + ih^* \\ f - ih & f^* - ih^* \end{pmatrix}.$$

They generalize the $N = 2$ sections

$$\mathcal{V}_{IJ} = \mathcal{V}\varepsilon_{IJ}\,, = \begin{pmatrix} \mathcal{L}^{\Lambda}\varepsilon_{IJ} \\ \mathcal{M}_{\Lambda}\varepsilon_{IJ} \end{pmatrix}, \qquad \text{and} \qquad \mathcal{V}_{i} = \mathcal{D}_{i}\mathcal{V} = \begin{pmatrix} f^{\Lambda}{}_{i} \\ h_{\Lambda\, i} \end{pmatrix}.$$

The graviphotons $A^{IJ}{}_{\mu}$ do not appear directly, only through the “dressed” vectors

$$A^{\Lambda}{}_{\mu} \equiv \tfrac{1}{2} f^{\Lambda}{}_{IJ} A^{IJ}{}_{\mu} + f^{\Lambda}{}_{i} A^{i}{}_{\mu}\,.$$

The supersymmetry transformations of the fermioninc fields are

$$\begin{aligned}
\delta_\epsilon \psi_{I\mu} &= \mathfrak{D}_\mu \epsilon_I + T_{IJ}{}^{+}{}_{\mu\nu}\gamma^\nu \epsilon^J \,, \\
\delta_\epsilon \chi_{IJK} &= -\tfrac{3i}{2} \not{T}_{[IJ}{}^{+}\epsilon_{K]} + i \not{P}_{IJKL}\epsilon^L \,, \\
\delta_\epsilon \lambda_{iI} &= -\tfrac{i}{2} \not{T}_i{}^{+}\epsilon_I + i \not{P}_{iIJ}\epsilon^J \,, \\
\delta_\epsilon \chi_{IJKLM} &= -5i \not{P}_{[IJKL}\epsilon_{M]} + \tfrac{i}{2}\varepsilon_{IJKLMN} \not{T}^{-}\epsilon^N + \tfrac{i}{4}\varepsilon_{IJKLMNOP} \not{T}^{NO-}\epsilon^P \,, \\
\delta_\epsilon \lambda_{iIJK} &= -3i \not{P}_{i[IJ}\epsilon_{K]} + \tfrac{i}{2}\varepsilon_{IJKL} \not{T}_i{}^{-}\epsilon^L + \tfrac{i}{4}\varepsilon_{IJKLMN} \not{T}^{LM-}\epsilon_N \,,
\end{aligned}$$

The supersymmetry transformations of the fermioninc fields are

$$\delta_\epsilon \psi_{I\mu} = \mathfrak{D}_\mu \epsilon_I + T_{IJ}{}^{+}{}_{\mu\nu}\gamma^\nu \epsilon^J \,,$$

$$\delta_\epsilon \chi_{IJK} = -\tfrac{3i}{2}\, \not{T}_{[IJ}{}^{+}\epsilon_{K]} + i\, \not{P}_{IJKL}\epsilon^L \,,$$

$$\delta_\epsilon \lambda_{iI} = -\tfrac{i}{2}\, \not{T}_i{}^{+}\epsilon_I + i\, \not{P}_{iIJ}\epsilon^J \,,$$

$$\delta_\epsilon \chi_{IJKLM} = -5i\, \not{P}_{[IJKL}\epsilon_{M]} + \tfrac{i}{2}\varepsilon_{IJKLMN}\, \not{T}^{-}\epsilon^N + \tfrac{i}{4}\varepsilon_{IJKLMNOP}\, \not{T}^{NO-}\epsilon^P \,,$$

$$\delta_\epsilon \lambda_{iIJK} = -3i\, \not{P}_{i[IJ}\epsilon_{K]} + \tfrac{i}{2}\varepsilon_{IJKL}\, \not{T}_i{}^{-}\epsilon^L + \tfrac{i}{4}\varepsilon_{IJKLMN}\, \not{T}^{LM-}\epsilon_N \,,$$

where the graviphoton and matter vector field strengths are

$$T_{IJ}{}^{+} = \langle\, \mathcal{V}_{IJ} \mid \mathcal{F}^{+}\,\rangle \,, \quad T_i{}^{+} = \langle\, \mathcal{V}_i \mid \mathcal{F}^{+}\,\rangle \,, \quad \mathcal{F}_\Lambda{}^{+} = \mathcal{N}^*_{\Lambda\Sigma} F^{\Sigma\,+} \,,$$

The supersymmetry transformations of the fermioninc fields are

$$\begin{aligned}
\delta_\epsilon \psi_{I\mu} &= \mathfrak{D}_\mu \epsilon_I + T_{IJ}{}^{+}{}_{\mu\nu}\gamma^\nu \epsilon^J \,, \\
\delta_\epsilon \chi_{IJK} &= -\tfrac{3i}{2}\, \not{T}_{[IJ}{}^{+}\epsilon_{K]} + i\, \not{P}_{IJKL}\epsilon^L \,, \\
\delta_\epsilon \lambda_{iI} &= -\tfrac{i}{2}\, \not{T}_i{}^{+}\epsilon_I + i\, \not{P}_{iIJ}\epsilon^J \,, \\
\delta_\epsilon \chi_{IJKLM} &= -5i\, \not{P}_{[IJKL}\epsilon_{M]} + \tfrac{i}{2}\varepsilon_{IJKLMN}\, \not{T}^{-}\epsilon^N + \tfrac{i}{4}\varepsilon_{IJKLMNOP}\, \not{T}^{NO-}\epsilon^P \,, \\
\delta_\epsilon \lambda_{iIJK} &= -3i\, \not{P}_{i[IJ}\epsilon_{K]} + \tfrac{i}{2}\varepsilon_{IJKL}\, \not{T}_i{}^{-}\epsilon^L + \tfrac{i}{4}\varepsilon_{IJKLMN}\, \not{T}^{LM-}\epsilon_N \,,
\end{aligned}$$

where the graviphoton and matter vector field strengths are

$$T_{IJ}{}^{+} = \langle\, \mathcal{V}_{IJ} \mid \mathcal{F}^{+}\,\rangle \,, \quad T_i{}^{+} = \langle\, \mathcal{V}_i \mid \mathcal{F}^{+}\,\rangle \,, \quad \mathcal{F}_\Lambda{}^{+} = \mathcal{N}^*_{\Lambda\Sigma} F^{\Sigma\,+} \,,$$

and where

$$\mathfrak{D}_\mu \epsilon_I \equiv \nabla_\mu \epsilon_I - \epsilon_J \Omega_\mu{}^J{}_I \,,$$

and $\Omega_\mu{}^J{}_I$ is the pullback of the connection of the scalar manifold ($\subset U(N)$).

The action for the bosonic fields is

$$S = \int d^4x \sqrt{|g|} \left[R + 2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda\,\mu\nu} F^{\Sigma}{}_{\mu\nu} - 2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda\,\mu\nu} \star F^{\Sigma}{}_{\mu\nu} \right.$$

$$\left. + \tfrac{2}{4!}\alpha_1 P^{*\,IJKL}{}_{\mu} P_{IJKL}{}^{\mu} + \alpha_2 P^{*\,iIJ}{}_{\mu} P_{iIJ}{}^{\mu} \right] ,$$

where

$$\mathcal{N} = hf^{-1} = \mathcal{N}^T , \qquad h_\Lambda = \mathcal{N}_{\Lambda\Sigma} f^\Sigma . \qquad \mathfrak{D} h_\Lambda = \mathcal{N}^*_{\Lambda\Sigma} \mathfrak{D} f^\Lambda .$$

The action for the bosonic fields is

$$S = \int d^4x\sqrt{|g|}\,\Big[R + 2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}F^{\Sigma}{}_{\mu\nu} - 2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}\star F^{\Sigma}{}_{\mu\nu}$$

$$+\tfrac{2}{4!}\alpha_1 P^{*\,IJKL}{}_{\mu}P_{IJKL}{}^{\mu} + \alpha_2 P^{*\,iIJ}{}_{\mu}P_{iIJ}{}^{\mu}\Big]\,,$$

where

$$\mathcal{N} = hf^{-1} = \mathcal{N}^T\,, \qquad h_\Lambda = \mathcal{N}_{\Lambda\Sigma}f^{\Sigma}\,. \qquad \mathfrak{D}h_\Lambda = \mathcal{N}^*_{\Lambda\Sigma}\mathfrak{D}f^{\Lambda}\,.$$

The N-specific constraints must be taken into account to find the e.o.m.:

The action for the bosonic fields is

$$S = \int d^4x \sqrt{|g|} \left[R + 2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda\,\mu\nu} F^{\Sigma}{}_{\mu\nu} - 2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda\,\mu\nu} \star F^{\Sigma}{}_{\mu\nu} \right.$$

$$\left. + \tfrac{2}{4!}\alpha_1 P^{*\,IJKL}{}_{\mu} P_{IJKL}{}^{\mu} + \alpha_2 P^{*\,iIJ}{}_{\mu} P_{iIJ}{}^{\mu} \right] ,$$

where

$$\mathcal{N} = hf^{-1} = \mathcal{N}^T , \qquad h_\Lambda = \mathcal{N}_{\Lambda\Sigma} f^\Sigma . \qquad \mathfrak{D} h_\Lambda = \mathcal{N}^*_{\Lambda\Sigma} \mathfrak{D} f^\Lambda .$$

The N-specific constraints must be taken into account to find the e.o.m.:

For $N = 2$ **:** $\mathcal{E}^{iIJ} = \mathfrak{D}^\mu P^{*\,iIJ}{}_\mu + 2T^{i\,-}{}_{\mu\nu} T^{IJ\,-\,\mu\nu} + P^{*\,iIJ\,A} P^{*\,jk}{}_A T_j{}^{+}{}_{\mu\nu} T_k{}^{+\,\mu\nu} .$

The action for the bosonic fields is

$$S = \int d^4x \sqrt{|g|}\, \left[R + 2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda\,\mu\nu} F^{\Sigma}{}_{\mu\nu} - 2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda\,\mu\nu} \star F^{\Sigma}{}_{\mu\nu} \right.$$

$$\left. + \tfrac{2}{4!}\alpha_1 P^{*\,IJKL}{}_{\mu} P_{IJKL}{}^{\mu} + \alpha_2 P^{*\,iIJ}{}_{\mu} P_{iIJ}{}^{\mu}\right] ,$$

where

$$\mathcal{N} = hf^{-1} = \mathcal{N}^T , \qquad h_\Lambda = \mathcal{N}_{\Lambda\Sigma} f^{\Sigma} . \qquad \mathfrak{D} h_\Lambda = \mathcal{N}^*_{\Lambda\Sigma} \mathfrak{D} f^{\Lambda} .$$

The N-specific constraints must be taken into account to find the e.o.m.:

For $N = 2$ **:** $\mathcal{E}^{iIJ} = \mathfrak{D}^{\mu} P^{*\,iIJ}{}_{\mu} + 2T^{i\,-}{}_{\mu\nu} T^{IJ\,-\,\mu\nu} + P^{*\,iIJ\,A} P^{*\,jk}{}_{A} T_{j}{}^{+}{}_{\mu\nu} T_{k}{}^{+\,\mu\nu}$.

For $N = 3$ **:** $\mathcal{E}^{iIJ} = \mathfrak{D}^{\mu} P^{*\,iIJ}{}_{\mu} + 2T^{i\,-}{}_{\mu\nu} T^{IJ\,-\,\mu\nu}$.

The action for the bosonic fields is

$$S = \int d^4x\sqrt{|g|}\,\left[R + 2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}F^{\Sigma}{}_{\mu\nu} - 2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}\star F^{\Sigma}{}_{\mu\nu}\right.$$

$$\left. + \tfrac{2}{4!}\alpha_1 P^{*\,IJKL}{}_{\mu}P_{IJKL}{}^{\mu} + \alpha_2 P^{*\,iIJ}{}_{\mu}P_{iIJ}{}^{\mu}\right]\,,$$

where

$$\mathcal{N} = hf^{-1} = \mathcal{N}^{T}\,, \qquad h_{\Lambda} = \mathcal{N}_{\Lambda\Sigma}f^{\Sigma}\,. \qquad \mathfrak{D}h_{\Lambda} = \mathcal{N}^{*}_{\Lambda\Sigma}\mathfrak{D}f^{\Lambda}\,.$$

The N-specific constraints must be taken into account to find the e.o.m.:

For $N=2$ **:** $\mathcal{E}^{iIJ} = \mathfrak{D}^{\mu}P^{*\,iIJ}{}_{\mu} + 2T^{i\,-}{}_{\mu\nu}T^{IJ\,-\,\mu\nu} + P^{*\,iIJ\,A}P^{*\,jk}{}_{A}T_{j}{}^{+}{}_{\mu\nu}T_{k}{}^{+\,\mu\nu}$.

For $N=3$ **:** $\mathcal{E}^{iIJ} = \mathfrak{D}^{\mu}P^{*\,iIJ}{}_{\mu} + 2T^{i\,-}{}_{\mu\nu}T^{IJ\,-\,\mu\nu}$.

For $N=4$ **:**
$$\left\{\begin{array}{rcl} \mathcal{E}^{IJKL} & = & \mathfrak{D}^{\mu}P^{*\,IJKL}{}_{\mu} + 6T^{[IJ|-}{}_{\mu\nu}T^{|KL]-\,\mu\nu} \\ & & + P^{*\,IJKL\,A}P^{*\,ij}{}_{A}T_{i}{}^{+}{}_{\mu\nu}T_{j}{}^{+\,\mu\nu}\,, \\ \mathcal{E}^{iIJ} & = & \mathfrak{D}^{\mu}P^{*\,iIJ}{}_{\mu} + T^{i\,-}{}_{\mu\nu}T^{IJ\,-\,\mu\nu} + \tfrac{1}{2}\varepsilon^{IJKL}T_{i}{}^{+}{}_{\mu\nu}T_{KL}{}^{+\,\mu\nu}\,. \end{array}\right.$$

The action for the bosonic fields is

$$S = \int d^4x\sqrt{|g|}\,\Big[R + 2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}F^{\Sigma}{}_{\mu\nu} - 2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}\star F^{\Sigma}{}_{\mu\nu}$$

$$+\tfrac{2}{4!}\alpha_1 P^{*\,IJKL}{}_{\mu}P_{IJKL}{}^{\mu} + \alpha_2 P^{*\,iIJ}{}_{\mu}P_{iIJ}{}^{\mu}\Big]\,,$$

where

$$\mathcal{N} = hf^{-1} = \mathcal{N}^{T}\,, \qquad h_{\Lambda} = \mathcal{N}_{\Lambda\Sigma}f^{\Sigma}\,. \qquad \mathfrak{D}h_{\Lambda} = \mathcal{N}^{*}_{\Lambda\Sigma}\mathfrak{D}f^{\Lambda}\,.$$

The N-specific constraints must be taken into account to find the e.o.m.:

For $N=2$: $\mathcal{E}^{iIJ} = \mathfrak{D}^{\mu}P^{*\,iIJ}{}_{\mu} + 2T^{i\,-}{}_{\mu\nu}T^{IJ\,-\,\mu\nu} + P^{*\,iIJ\,A}P^{*\,jk}{}_{A}T_{j}{}^{+}{}_{\mu\nu}T_{k}{}^{+\,\mu\nu}$.

For $N=3$: $\mathcal{E}^{iIJ} = \mathfrak{D}^{\mu}P^{*\,iIJ}{}_{\mu} + 2T^{i\,-}{}_{\mu\nu}T^{IJ\,-\,\mu\nu}$.

For $N=4$:

$$\begin{cases} \mathcal{E}^{IJKL} = \mathfrak{D}^{\mu}P^{*\,IJKL}{}_{\mu} + 6T^{[IJ|-}{}_{\mu\nu}T^{|KL]-\,\mu\nu} \\ \qquad\qquad +P^{*\,IJKL\,A}P^{*\,ij}{}_{A}T_{i}{}^{+}{}_{\mu\nu}T_{j}{}^{+\,\mu\nu}\,, \\ \mathcal{E}^{iIJ} = \mathfrak{D}^{\mu}P^{*\,iIJ}{}_{\mu} + T^{i\,-}{}_{\mu\nu}T^{IJ\,-\,\mu\nu} + \tfrac{1}{2}\varepsilon^{IJKL}T_{i}{}^{+}{}_{\mu\nu}T_{KL}{}^{+\,\mu\nu}\,. \end{cases}$$

For $N=5$: $\mathcal{E}^{IJKL} = \mathfrak{D}^{\mu}P^{*\,IJKL}{}_{\mu} + 6T^{[IJ|-}{}_{\mu\nu}T^{|KL]-\,\mu\nu}$. etc.

8 – The all-N Killing Spinor Equations (KSEs)

For all values of N the independent KSEs take the form

$$
\begin{aligned}
\mathfrak{D}_\mu \epsilon_I + T_{IJ}{}^{+}{}_{\mu\nu}\gamma^\nu \epsilon^J &= 0\,, \\
\not{P}_{IJKL}\epsilon^L - \tfrac{3}{2}\not{T}_{[IJ}{}^{+}\epsilon_{K]} &= 0\,, \\
\not{P}_{i\,IJ}\epsilon^J - \tfrac{1}{2}\not{T}_i{}^{+}\epsilon_I &= 0\,, \\
\not{P}_{[IJKL}\epsilon_{M]} &= 0\,, \\
\not{P}_{i\,[IJ}\epsilon_{K]} &= 0\,.
\end{aligned}
$$

The last two KSEs should only be considered for $N = 5$ and $N = 3$, resp.

8 – The all-N Killing Spinor Equations (KSEs)

For all values of N the independent KSEs take the form

$$
\begin{aligned}
\mathfrak{D}_\mu \epsilon_I + T_{IJ}{}^{+}{}_{\mu\nu}\gamma^\nu \epsilon^J &= 0\,,\\
\not{P}_{IJKL}\epsilon^L - \tfrac{3}{2}\not{T}_{[IJ}{}^{+}\epsilon_{K]} &= 0\,,\\
\not{P}_{i\,IJ}\epsilon^J - \tfrac{1}{2}\not{T}_i{}^{+}\epsilon_I &= 0\,,\\
\not{P}_{[IJKL}\epsilon_{M]} &= 0\,,\\
\not{P}_{i\,[IJ}\epsilon_{K]} &= 0\,.
\end{aligned}
$$

The last two KSEs should only be considered for $N = 5$ and $N = 3$, resp.

Again, our goal is to find **all** the bosonic field configurations $\{e^a{}_\mu, A^\Lambda{}_\mu, P_{IJKL\,\mu}, P_{i\,IJ\,\mu}\}$ such that the above KSEs admit at least one solution ϵ^I.

9 – The all-N spinor-bilinears algebra

The independent bilinears that we can construct with one $U(N)$ vector of Weyl spinors ϵ_I are:

9 – The all-N spinor-bilinears algebra

The independent bilinears that we can construct with one $U(N)$ vector of Weyl spinors ϵ_I are:

1. A complex antisymmetric matrix of scalars $M_{IJ} \equiv \bar{\epsilon}_I \epsilon_J = -M_{JI}$.

9 – The all-N spinor-bilinears algebra

The independent bilinears that we can construct with one $U(N)$ vector of Weyl spinors ϵ_I are:

1. A complex antisymmetric matrix of scalars $M_{IJ} \equiv \bar{\epsilon}_I \epsilon_J = -M_{JI}$.
2. A Hermitean matrix of vectors $V^I{}_{J\,a} \equiv i\bar{\epsilon}^I \gamma_a \epsilon_J$.

9 – The all-N spinor-bilinears algebra

The independent bilinears that we can construct with one $U(N)$ vector of Weyl spinors ϵ_I are:

1. A complex antisymmetric matrix of scalars $M_{IJ} \equiv \bar{\epsilon}_I \epsilon_J = -M_{JI}$.
2. A Hermitean matrix of vectors $V^I{}_{J\,a} \equiv i\bar{\epsilon}^I \gamma_a \epsilon_J$.

The 4- Fierz identities imply the following properties for them:

9 – The all-N spinor-bilinears algebra

The independent bilinears that we can construct with one $U(N)$ vector of Weyl spinors ϵ_I are:

1. A complex antisymmetric matrix of scalars $M_{IJ} \equiv \bar{\epsilon}_I \epsilon_J = -M_{JI}$.
2. A Hermitean matrix of vectors $V^I{}_{J\,a} \equiv i\bar{\epsilon}^I \gamma_a \epsilon_J$.

The 4- Fierz identities imply the following properties for them:

1. $M_{I[J} M_{KL]} = 0$, so rank$\,(M_{IJ}) \leq 2$.

9 – The all-N spinor-bilinears algebra

The independent bilinears that we can construct with one $U(N)$ vector of Weyl spinors ϵ_I are:

1. A complex antisymmetric matrix of scalars $M_{IJ} \equiv \bar{\epsilon}_I \epsilon_J = -M_{JI}$.
2. A Hermitean matrix of vectors $V^I{}_{J\,a} \equiv i\bar{\epsilon}^I \gamma_a \epsilon_J$.

The 4- Fierz identities imply the following properties for them:

1. $M_{I[J} M_{KL]} = 0$, so rank $(M_{IJ}) \leq 2$.
2. $V_a \equiv V^I{}_{I\,a}$ is always non-spacelike: $V^2 = 2M^{IJ} M_{IJ} \equiv 2|M|^2 \geq 0$.

9 – The all-N spinor-bilinears algebra

The independent bilinears that we can construct with one $U(N)$ vector of Weyl spinors ϵ_I are:

1. A complex antisymmetric matrix of scalars $M_{IJ} \equiv \bar{\epsilon}_I \epsilon_J = -M_{JI}$.
2. A Hermitean matrix of vectors $V^I{}_{J\,a} \equiv i\bar{\epsilon}^I \gamma_a \epsilon_J$.

The 4- Fierz identities imply the following properties for them:

1. $M_{I[J}M_{KL]} = 0$, so rank $(M_{IJ}) \leq 2$.
2. $V_a \equiv V^I{}_{I\,a}$ is always non-spacelike: $V^2 = 2M^{IJ}M_{IJ} \equiv 2|M|^2 \geq 0$.
   We only consider the timelike case.

9 – The all-N spinor-bilinears algebra

The independent bilinears that we can construct with one $U(N)$ vector of Weyl spinors ϵ_I are:

1. A complex antisymmetric matrix of scalars $M_{IJ} \equiv \bar{\epsilon}_I \epsilon_J = -M_{JI}$.
2. A Hermitean matrix of vectors $V^I{}_{J\,a} \equiv i\bar{\epsilon}^I \gamma_a \epsilon_J$.

The 4- Fierz identities imply the following properties for them:

1. $M_{I[J}M_{KL]} = 0$, so rank $(M_{IJ}) \leq 2$.
2. $V_a \equiv V^I{}_{I\,a}$ is always non-spacelike: $V^2 = 2M^{IJ}M_{IJ} \equiv 2|M|^2 \geq 0$.
   We only consider the timelike case.
3. We can choose a tetrad $\{e^a{}_\mu\}$ such that $e^0{}_\mu \equiv \frac{1}{\sqrt{2}}|M|^{-1}V_\mu$. Then, **defining** $V^m{}_\mu \equiv |M|e^m{}_\mu$ we can decompose

$$V^I{}_{J\,\mu} = \tfrac{1}{2}\mathcal{J}^I{}_J V_\mu + \tfrac{1}{\sqrt{2}}(\sigma^m)^I{}_J V^m{}_\mu\,,$$

   where $\mathcal{J}^I{}_J = 2M^{IK}M_{JK}|M|^{-2}$ is a rank 2 projector (Tod):

$$\mathcal{J}^2 = \mathcal{J}\,, \qquad \mathcal{J}^I{}_I = +2\,, \qquad \mathcal{J}^I{}_J\epsilon^J = \epsilon^I\,.$$

The main properties satisfied by the three σ^m matrices are:

$$
\begin{aligned}
\sigma^m \sigma^n &= \delta^{mn}\mathcal{J} + i\varepsilon^{mnp}\sigma^p\,, \\
\mathcal{J}\sigma^m &= \sigma^m\mathcal{J} = \sigma^m\,, \\
(\sigma^m)^I{}_I &= 0\,, \\
\mathcal{J}^K{}_J\mathcal{J}^L{}_I &= \tfrac{1}{2}\mathcal{J}^K{}_I\mathcal{J}^L{}_J + \tfrac{1}{2}(\sigma^m)^K{}_I(\sigma^m)^L{}_J\,, \\
M_{K[I}(\sigma^m)^K{}_{J]} &= 0\,, \\
2|M|^{-2}M_{LI}(\sigma^m)^I{}_J M^{JK} &= (\sigma^m)^K{}_L\,,
\end{aligned}
$$

The main properties satisfied by the three σ^m matrices are:

$$\begin{aligned}
\sigma^m \sigma^n &= \delta^{mn} \mathcal{J} + i\varepsilon^{mnp}\sigma^p \,, \\
\mathcal{J}\sigma^m &= \sigma^m \mathcal{J} = \sigma^m \,, \\
(\sigma^m)^I{}_I &= 0 \,, \\
\mathcal{J}^K{}_J \mathcal{J}^L{}_I &= \tfrac{1}{2}\mathcal{J}^K{}_I \mathcal{J}^L{}_J + \tfrac{1}{2}(\sigma^m)^K{}_I (\sigma^m)^L{}_J \,, \\
M_{K[I}(\sigma^m)^K{}_{J]} &= 0 \,, \\
2|M|^{-2} M_{LI}(\sigma^m)^I{}_J M^{JK} &= (\sigma^m)^K{}_L \,,
\end{aligned}$$

$\{\mathcal{J}, \sigma^1, \sigma^2, \sigma^3\}$ is an x-dependent basis of a $\mathfrak{u}(2)$ subalgebra of $\mathfrak{u}(N)$ in the 2-dimensional eigenspace of $\mathcal{J}$ of eigenvalue $+1$ and provide a basis in the space of Hermitean matrices A satisfying $\mathcal{J} A \mathcal{J} = A$

10 – The all-N Killing Spinor Identities (KSIs)

If we assume that a given bosonic field configuration admits a Killing spinor ϵ_I, then we find that the (*off-shell*) "equations of motion" $\{\mathcal{E}^{\mu\nu}, \mathcal{E}^{\mu}, \mathcal{E}^{IJKL}, \mathcal{E}^{i\,IJ}\}$ satisfy the KSIs ($\tilde{\mathcal{J}}^{I}{}_{J} \equiv \delta^{I}{}_{J} - \mathcal{J}^{I}{}_{J}$):

10 – The all-N Killing Spinor Identities (KSIs)

If we assume that a given bosonic field configuration admits a Killing spinor ϵ_I, then we find that the (*off-shell*) "equations of motion" $\{\mathcal{E}^{\mu\nu}, \mathcal{E}^{\mu}, \mathcal{E}^{IJKL}, \mathcal{E}^{i\,IJ}\}$ satisfy the KSIs ($\tilde{\mathcal{J}}^I{}_J \equiv \delta^I{}_J - \mathcal{J}^I{}_J$):

1. $\mathcal{E}^{0m} = \mathcal{E}^{mn} = 0.$

10 – The all-N Killing Spinor Identities (KSIs)

If we assume that a given bosonic field configuration admits a Killing spinor ϵ_I, then we find that the (*off-shell*) "equations of motion" $\{\mathcal{E}^{\mu\nu}, \mathcal{E}^{\mu}, \mathcal{E}^{IJKL}, \mathcal{E}^{i\,IJ}\}$ satisfy the KSIs ($\tilde{\mathcal{J}}^I{}_J \equiv \delta^I{}_J - \mathcal{J}^I{}_J$):

1. $\mathcal{E}^{0m} = \mathcal{E}^{mn} = 0.$

2. $\mathcal{E}^{m} = 0.$

10 – The all-N Killing Spinor Identities (KSIs)

If we assume that a given bosonic field configuration admits a Killing spinor ϵ_I, then we find that the (*off-shell*) "equations of motion" $\{\mathcal{E}^{\mu\nu}, \mathcal{E}^{\mu}, \mathcal{E}^{IJKL}, \mathcal{E}^{i\,IJ}\}$ satisfy the KSIs ($\tilde{\mathcal{J}}^I{}_J \equiv \delta^I{}_J - \mathcal{J}^I{}_J$):

1. $\mathcal{E}^{0m} = \mathcal{E}^{mn} = 0.$

2. $\mathcal{E}^{m} = 0.$

3. $$\left\{ \begin{array}{rcl} \mathcal{E}^{MNPQ}\mathcal{J}^{[I}{}_M\tilde{\mathcal{J}}^J{}_N\tilde{\mathcal{J}}^K{}_P\tilde{\mathcal{J}}^{L]}{}_Q & = & 0\,, \\ \mathcal{E}^{i\,MN}\mathcal{J}^{[I}{}_M\tilde{\mathcal{J}}^{J]}{}_N & = & 0\,, \end{array} \right.$$ ($\Rightarrow$ no attractor mechanism)

10 – The all-N Killing Spinor Identities (KSIs)

If we assume that a given bosonic field configuration admits a Killing spinor ϵ_I, then we find that the (*off-shell*) "equations of motion" $\{\mathcal{E}^{\mu\nu}, \mathcal{E}^{\mu}, \mathcal{E}^{IJKL}, \mathcal{E}^{i\,IJ}\}$ satisfy the KSIs ($\tilde{\mathcal{J}}^{I}{}_{J} \equiv \delta^{I}{}_{J} - \mathcal{J}^{I}{}_{J}$):

1. $\mathcal{E}^{0m} = \mathcal{E}^{mn} = 0.$

2. $\mathcal{E}^{m} = 0.$

3. $$\left\{ \begin{array}{rcl} \mathcal{E}^{MNPQ}\mathcal{J}^{[I}{}_{M}\tilde{\mathcal{J}}^{J}{}_{N}\tilde{\mathcal{J}}^{K}{}_{P}\tilde{\mathcal{J}}^{L]}{}_{Q} & = & 0\,, \\ \mathcal{E}^{i\,MN}\mathcal{J}^{[I}{}_{M}\tilde{\mathcal{J}}^{J]}{}_{N} & = & 0\,, \end{array} \right.$$ ($\Rightarrow$ no attractor mechanism)

4. $\mathcal{E}^{00} = -2\sqrt{2}\langle\,\mathcal{E}^{0} \mid \Re\mathrm{e}\left(\mathcal{V}_{IJ}\frac{M^{IJ}}{|M|}\right)\rangle$, (Bogomol'nyi bound)

5. $\langle \mathcal{E}^0 \mid \Im\mathrm{m} \left(\mathcal{V}_{IJ} \frac{M^{IJ}}{|M|} \right) \rangle$, ($\Rightarrow$ no NUT charge).

5. $\langle\, \mathcal{E}^0 \mid \Im\mathrm{m}\left(\mathcal{V}_{IJ} \frac{M^{IJ}}{|M|}\right) \rangle$, ($\Rightarrow$ no NUT charge).

6. $\left\{ \begin{array}{l} \mathcal{E}^{MNPQ} \mathcal{J}^{[I}{}_M \mathcal{J}^{J}{}_N \tilde{\mathcal{J}}^{K}{}_P \tilde{\mathcal{J}}^{L]}{}_Q \,, \\ \\ \mathcal{E}^{i\,MN} \mathcal{J}^{[I}{}_M \mathcal{J}^{J]}{}_N \,, \end{array} \right.$ are related to $\mathcal{E}^0$ ($\Rightarrow$ attractor mechanism)

5. $\langle \mathcal{E}^0 \mid \Im\mathrm{m}\left(\mathcal{V}_{IJ}\frac{M^{IJ}}{|M|}\right)\rangle$, ($\Rightarrow$ no NUT charge).

6. $\left\{ \begin{array}{l} \mathcal{E}^{MNPQ}\mathcal{J}^{[I}{}_{M}\mathcal{J}^{J}{}_{N}\tilde{\mathcal{J}}^{K}{}_{P}\tilde{\mathcal{J}}^{L]}{}_{Q}\,, \\ \\ \mathcal{E}^{i\,MN}\mathcal{J}^{[I}{}_{M}\mathcal{J}^{J]}{}_{N}\,, \end{array}\right.$ are related to $\mathcal{E}^0$ ($\Rightarrow$ attractor mechanism)

The precise form of the relation depends on N:

5. $\langle \mathcal{E}^0 \mid \Im\mathrm{m}\left(\mathcal{V}_{IJ}\frac{M^{IJ}}{|M|}\right)\rangle$, ($\Rightarrow$ no NUT charge).

6. $\left\{\begin{array}{l}\mathcal{E}^{MNPQ}\mathcal{J}^{[I}{}_M\mathcal{J}^{J}{}_N\tilde{\mathcal{J}}^{K}{}_P\tilde{\mathcal{J}}^{L]}{}_Q\,,\\ \mathcal{E}^{i\,MN}\mathcal{J}^{[I}{}_M\mathcal{J}^{J]}{}_N\,,\end{array}\right.$ are related to $\mathcal{E}^0$ ($\Rightarrow$ attractor mechanism)

The precise form of the relation depends on N:

$$N=3:\ \mathcal{E}^{i\,IJ} = -2\sqrt{2}\frac{M^{IJ}}{|M|}\langle\,\mathcal{E}^0 \mid \mathcal{V}^{*\,i}\,\rangle\,,$$

5. $\langle\, \mathcal{E}^0 \mid \Im\mathrm{m} \left(\mathcal{V}_{IJ} \frac{M^{IJ}}{|M|} \right) \rangle$, ($\Rightarrow$ no NUT charge).

6. $\left\{ \begin{array}{l} \mathcal{E}^{MNPQ} \mathcal{J}^{[I}{}_{M} \mathcal{J}^{J}{}_{N} \tilde{\mathcal{J}}^{K}{}_{P} \tilde{\mathcal{J}}^{L]}{}_{Q}\,, \\ \mathcal{E}^{i\,MN} \mathcal{J}^{[I}{}_{M} \mathcal{J}^{J]}{}_{N}\,, \end{array} \right.$ are related to $\mathcal{E}^0$ ($\Rightarrow$ attractor mechanism)

The precise form of the relation depends on N:

$$N = 3:\ \mathcal{E}^{i\,IJ} = -2\sqrt{2}\frac{M^{IJ}}{|M|}\langle\, \mathcal{E}^0 \mid \mathcal{V}^{*\,i}\,\rangle\,,$$

$$N = 4:\ \left\{ \begin{array}{rcl} \mathcal{E}^{IJKL} & = & -2\sqrt{2}\frac{M^{[IJ|}}{|M|}\langle\, \mathcal{E}^0 \mid \mathcal{V}^{*\,|KL]}\,\rangle\,, \\ \mathcal{E}_{iIJ} & = & -2\sqrt{2}\left\{ \frac{M_{IJ}}{|M|}\langle\, \mathcal{E}^0 \mid \mathcal{V}_i\,\rangle + \frac{1}{2}\varepsilon_{IJKL}\frac{M^{KL}}{|M|}\langle\, \mathcal{E}^0 \mid \mathcal{V}^{*\,i}\,\rangle \right\}\,, \end{array} \right.$$

etc.

The only independent equations of motion that have to be imposed on **any** $d = 4$ supersymmetric configuration are

$$\mathcal{E}^0 = 0\,.$$

11 – The all-N supersymmetric solutions

The construction of any timelike supersymmetric solution proceeds as follows:

11 – The all-N supersymmetric solutions

The construction of any timelike supersymmetric solution proceeds as follows:

I. Choose the $U(2)$ subgroup determining the associated $N=2$ truncation:

11 – The all-N supersymmetric solutions

The construction of any timelike supersymmetric solution proceeds as follows:

I. Choose the $U(2)$ subgroup determining the associated $N = 2$ truncation:

1. Choose x-dependent rank-2, $N \times N$ complex antisymmetric M_{IJ}. With it we construct the projector $\mathcal{J}^I{}_J \equiv 2|M|^{-2} M^{IK} M_{JK}$.

11 – The all-N supersymmetric solutions

The construction of any timelike supersymmetric solution proceeds as follows:

I. Choose the $U(2)$ subgroup determining the associated $N = 2$ truncation:

1. Choose x-dependent rank-2, $N \times N$ complex antisymmetric M_{IJ}. With it we construct the projector $\mathcal{J}^I{}_J \equiv 2|M|^{-2} M^{IK} M_{JK}$.
   Supersymmetry requires is covariant constancy

$$\mathfrak{D}\mathcal{J} \equiv d\mathcal{J} - [\mathcal{J}, \Omega] = 0\,,$$

   which implies constancy for $N = 2$, $N = 3$ and $N = 4$, but not in general.

11 – The all-N supersymmetric solutions

The construction of any timelike supersymmetric solution proceeds as follows:

I. Choose the $U(2)$ subgroup determining the associated $N = 2$ truncation:

1. Choose x-dependent rank-2, $N \times N$ complex antisymmetric M_{IJ}. With it we construct the projector $\mathcal{J}^I{}_J \equiv 2|M|^{-2} M^{IK} M_{JK}$.
   Supersymmetry requires is covariant constancy
   $$\mathfrak{D}\mathcal{J} \equiv d\mathcal{J} - [\mathcal{J}, \Omega] = 0\,,$$
   which implies constancy for $N = 2$, $N = 3$ and $N = 4$, but not in general.
2. Choose three $N \times N$, Hermitean , traceless, x-dependent $(\sigma^m)^I{}_J$, satisfying the same properties as the Pauli matrices in the subspace preserved by $\mathcal{J}$.

11 – The all-N supersymmetric solutions

The construction of any timelike supersymmetric solution proceeds as follows:

I. Choose the $U(2)$ subgroup determining the associated $N=2$ truncation:

1. Choose x-dependent rank-2, $N \times N$ complex antisymmetric M_{IJ}. With it we construct the projector $\mathcal{J}^I{}_J \equiv 2|M|^{-2} M^{IK} M_{JK}$.
   Supersymmetry requires is covariant constancy

   $$\mathfrak{D}\mathcal{J} \equiv d\mathcal{J} - [\mathcal{J}, \Omega] = 0\,,$$

   which implies constancy for $N=2$, $N=3$ and $N=4$, but not in general.

2. Choose three $N \times N$, Hermitean , traceless, x-dependent $(\sigma^m)^I{}_J$, satisfying the same properties as the Pauli matrices in the subspace preserved by $\mathcal{J}$.
   We also have to impose the constraint

   $$\mathcal{J} d\sigma^m \mathcal{J} = 0\,.$$

Once the $U(2)$ subgroup has been chosen, we can split the Vielbeins $P_{IJKL\,\mu}$ and $P_{i\,IJ\,\mu}$, into associated to the would-be vector multiplets in the $N=2$ truncation

$$P_{IJKL}\,\mathcal{J}^{I}{}_{[M}\mathcal{J}^{J}{}_{N}\tilde{\mathcal{J}}^{K}{}_{P}\tilde{\mathcal{J}}^{L}{}_{Q]}\,, \quad \text{and} \quad P_{i\,IJ}\,\mathcal{J}^{I}{}_{[K}\mathcal{J}^{J}{}_{L]}\,,$$

which are driven by the *attractor mechanism* (*i.e.* they are determined by the electric and magnetic charges) and those associated to the hypermultiplets

$$P_{IJKL}\,\mathcal{J}^{I}{}_{[M}\tilde{\mathcal{J}}^{J}{}_{N}\tilde{\mathcal{J}}^{K}{}_{P}\tilde{\mathcal{J}}^{L}{}_{Q]}\,, \quad \text{and} \quad P_{i\,IJ}\,\mathcal{J}^{I}{}_{[K}\tilde{\mathcal{J}}^{J}{}_{L]}\,.$$

which are not.

In hyper-less solutions (*e.g.* black holes) the σ^{m}s matrices are not needed at all.

II. After the choice of $U(2)$ subgroup, the solutions are constructed:

II. After the choice of $U(2)$ subgroup, the solutions are constructed:

1. Define the real symplectic vectors $\mathcal{R}$ and $\mathcal{I}$

$$\mathcal{R} + i\mathcal{I} \equiv |M|^{-2}\mathcal{V}_{IJ}M^{IJ} \, .$$

($U(N)$ singlets $\Rightarrow$ no $U(N)$ gauge -fixing necessary)

II. After the choice of $U(2)$ subgroup, the solutions are constructed:

1. Define the real symplectic vectors $\mathcal{R}$ and $\mathcal{I}$

$$\mathcal{R} + i\mathcal{I} \equiv |M|^{-2}\mathcal{V}_{IJ}M^{IJ}\,.$$

($U(N)$ singlets $\Rightarrow$ no $U(N)$ gauge -fixing necessary) **2.** The components of $\mathcal{I}$ are given by a symplectic vector real functions $\mathcal{H}$ harmonic in the 3-dimensional transverse space with metric $\gamma_{\underline{mn}}$:

II. After the choice of $U(2)$ subgroup, the solutions are constructed:

1. Define the real symplectic vectors $\mathcal{R}$ and $\mathcal{I}$

$$\mathcal{R} + i\mathcal{I} \equiv |M|^{-2}\mathcal{V}_{IJ}M^{IJ}\ .$$

($U(N)$ singlets $\Rightarrow$ no $U(N)$ gauge -fixing necessary) **2.** The components of $\mathcal{I}$ are given by a symplectic vector real functions $\mathcal{H}$ harmonic in the 3-dimensional transverse space with metric $\gamma_{\underline{mn}}$:

$$\boxed{\nabla^2_{(3)}\mathcal{H} = 0.}$$

II. After the choice of $U(2)$ subgroup, the solutions are constructed:

1. Define the real symplectic vectors $\mathcal{R}$ and $\mathcal{I}$

$$\mathcal{R} + i\mathcal{I} \equiv |M|^{-2}\mathcal{V}_{IJ}M^{IJ}\,.$$

($U(N)$ singlets $\Rightarrow$ no $U(N)$ gauge -fixing necessary) **2.** The components of $\mathcal{I}$ are given by a symplectic vector real functions $\mathcal{H}$ harmonic in the 3-dimensional transverse space with metric $\gamma_{\underline{mn}}$:

$$\boxed{\nabla^2_{(3)}\mathcal{H} = 0.}$$

3. $\mathcal{R}$ is to be be found from $\mathcal{I}$ by solving the generalized *stabilization equations*.

II. After the choice of $U(2)$ subgroup, the solutions are constructed:

1. Define the real symplectic vectors $\mathcal{R}$ and $\mathcal{I}$

$$\mathcal{R} + i\mathcal{I} \equiv |M|^{-2}\mathcal{V}_{IJ}M^{IJ}\,.$$

($U(N)$ singlets $\Rightarrow$ no $U(N)$ gauge -fixing necessary) **2.** The components of $\mathcal{I}$ are given by a symplectic vector real functions $\mathcal{H}$ harmonic in the 3-dimensional transverse space with metric $\gamma_{\underline{mn}}$:

$$\boxed{\nabla^2_{(3)}\mathcal{H} = 0.}$$

3. $\mathcal{R}$ is to be be found from $\mathcal{I}$ by solving the generalized *stabilization equations*.

4. The metric is

$$ds^2 \;=\; |M|^2(dt+\omega)^2 - |M|^{-2}\gamma_{\underline{mn}}dx^m dx^n\,.$$

II. After the choice of $U(2)$ subgroup, the solutions are constructed:

1. Define the real symplectic vectors $\mathcal{R}$ and $\mathcal{I}$

$$\mathcal{R} + i\mathcal{I} \equiv |M|^{-2}\mathcal{V}_{IJ}M^{IJ}\,.$$

($U(N)$ singlets $\Rightarrow$ no $U(N)$ gauge -fixing necessary) **2.** The components of $\mathcal{I}$ are given by a symplectic vector real functions $\mathcal{H}$ harmonic in the 3-dimensional transverse space with metric $\gamma_{\underline{mn}}$:

$$\boxed{\nabla^2_{(3)}\mathcal{H} = 0.}$$

3. $\mathcal{R}$ is to be be found from $\mathcal{I}$ by solving the generalized *stabilization equations*.

4. The metric is

$$ds^2 = |M|^2(dt+\omega)^2 - |M|^{-2}\gamma_{\underline{mn}}dx^m dx^n\,.$$

where

$$|M|^{-2} = (M^{IJ}M_{IJ})^{-2} = \langle\,\mathcal{R}\mid\mathcal{I}\,\rangle\,,$$

$$(d\omega)_{mn} = 2\epsilon_{mnp}\langle\,\mathcal{I}\mid\partial^p\mathcal{I}\,\rangle\,.$$

$\gamma_{\underline{mn}}$ is determined indirectly from the would-be hypers in the associated $N = 2$ truncation and its curvature vanishes when those scalars vanish.

$\gamma_{\underline{mn}}$ is determined indirectly from the would-be hypers in the associated $N = 2$ truncation and its curvature vanishes when those scalars vanish.

Its spin connection $\varpi^{mn}$is related to Ω, by

$$\varpi^{mn} = i\varepsilon^{mnp}\mathrm{Tr}\left[\sigma^p\Omega\right].$$

(Observe that only the $\mathfrak{su}(2)$ components of Ω constribute to ϖ^{mn}.

$\gamma_{\underline{mn}}$ is determined indirectly from the would-be hypers in the associated $N = 2$ truncation and its curvature vanishes when those scalars vanish.

Its spin connection $\varpi^{mn}$is related to Ω, by

$$\varpi^{mn} = i\varepsilon^{mnp}\mathrm{Tr}\left[\sigma^p\Omega\right].$$

(Observe that only the $\mathfrak{su}(2)$ components of Ω constribute to ϖ^{mn}.

5. The vector field strengths are

$$F = -\tfrac{1}{2}d(\mathcal{R}\hat{V}) - \tfrac{1}{2}\star(\hat{V}\wedge d\mathcal{I})\,, \qquad \hat{V} = \sqrt{2}|M|^2(dt+\omega)\,.$$

$\gamma_{\underline{mn}}$ is determined indirectly from the would-be hypers in the associated $N=2$ truncation and its curvature vanishes when those scalars vanish.

Its spin connection $\varpi^{mn}$is related to Ω, by

$$\varpi^{mn} = i\varepsilon^{mnp}\mathrm{Tr}\left[\sigma^p\Omega\right].$$

(Observe that only the $\mathfrak{su}(2)$ components of Ω constribute to ϖ^{mn}.

5. The vector field strengths are

$$F = -\tfrac{1}{2}d(\mathcal{R}\hat{V}) - \tfrac{1}{2}\star(\hat{V}\wedge d\mathcal{I})\,, \qquad \hat{V} = \sqrt{2}|M|^2(dt+\omega)\,.$$

6. The scalars in the vector multiplets in the associated $N=2$ truncation

$$P_{IJKL}\,\mathcal{J}^I{}_{[M}\mathcal{J}^J{}_N\tilde{\mathcal{J}}^K{}_P\tilde{\mathcal{J}}^L{}_{Q]}\,,\quad \text{and} \quad P_{i\,IJ}\,\mathcal{J}^I{}_{[K}\mathcal{J}^J{}_{L]}\,,$$

can be found from $\mathcal{R}$ and $\mathcal{I}$, while those in the hypers must be found independently by solving

$$\begin{aligned} P_{IJKL\,m}\,\mathcal{J}^I{}_{[M}\tilde{\mathcal{J}}^J{}_N\tilde{\mathcal{J}}^K{}_P\tilde{\mathcal{J}}^L{}_{Q]}(\sigma^m)^Q{}_R &= 0\,, \\ P_{i\,IJ\,m}\,\mathcal{J}^I{}_{[K}\tilde{\mathcal{J}}^J{}_{L]}(\sigma^m)^L{}_M &= 0\,, \end{aligned}$$

which solve their equations of motion according to the *Killing Spinor Identities*.

12 – Attractor flow equations

A simple derivation of the attractor flow eqs. in $N = 1, d = 5$ supergravity

12 – Attractor flow equations

A simple derivation of the attractor flow eqs. in $N = 1, d = 5$ supergravity

Consider $N = 1, d = 5$ supergravity coupled to n vector multiplets

$$\{A^x{}_\mu, \lambda^{ix}, \phi^x\}, \qquad x = 1, \cdots, n\,.$$

12 – Attractor flow equations

A simple derivation of the attractor flow eqs. in $N = 1, d = 5$ supergravity

Consider $N = 1, d = 5$ supergravity coupled to n vector multiplets

$$\{A^x{}_\mu, \lambda^{ix}, \phi^x\}, \qquad x = 1, \cdots, n\,.$$

The matter vector fields $A^x{}_\mu$ and the graviphoton $A^0{}_\mu$ are combined into an $SO(n+1)$ vector $A^I{}_\mu$ with $I = 0, x$.

12 – Attractor flow equations

A simple derivation of the attractor flow eqs. in $N=1, d=5$ supergravity

Consider $N=1, d=5$ supergravity coupled to n vector multiplets

$$\{A^{x}{}_{\mu}, \lambda^{ix}, \phi^{x}\}\,, \qquad\qquad x = 1, \cdots, n\,.$$

The matter vector fields $A^{x}{}_{\mu}$ and the graviphoton $A^{0}{}_{\mu}$ are combined into an $SO(n+1)$ vector $A^{I}{}_{\mu}$ with $I = 0, x$. To make manifest the symmetries, the n real scalars ϕ^{x} are described by $n+1$ functions $h^{I}(\phi)$ which are constrained:

$$C_{IJK} h^{I} h^{J} h^{K} = 1\,.$$

12 – Attractor flow equations

A simple derivation of the attractor flow eqs. in $N=1, d=5$ supergravity

Consider $N=1, d=5$ supergravity coupled to n vector multiplets

$$\{A^{x}{}_{\mu}, \lambda^{ix}, \phi^{x}\}, \qquad x = 1, \cdots, n\,.$$

The matter vector fields $A^{x}{}_{\mu}$ and the graviphoton $A^{0}{}_{\mu}$ are combined into an $SO(n+1)$ vector $A^{I}{}_{\mu}$ with $I=0,x$. To make manifest the symmetries, the n real scalars ϕ^{x} are described by $n+1$ functions $h^{I}(\phi)$ which are constrained:

$$C_{IJK}h^{I}h^{J}h^{K} = 1\,.$$

We introduce a function f and assume ($h_{I} \equiv C_{IJK}h^{J}h^{K}$)

$$h_{I}/f \equiv l_{I} + q_{I}\rho\,,$$

for some coordinate ρ.

12 – Attractor flow equations

A simple derivation of the attractor flow eqs. in $N = 1, d = 5$ supergravity

Consider $N = 1, d = 5$ supergravity coupled to n vector multiplets

$$\{A^x{}_\mu, \lambda^{ix}, \phi^x\}\,, \qquad x = 1, \cdots, n\,.$$

The matter vector fields $A^x{}_\mu$ and the graviphoton $A^0{}_\mu$ are combined into an $SO(n+1)$ vector $A^I{}_\mu$ with $I = 0, x$. To make manifest the symmetries, the n real scalars ϕ^x are described by $n+1$ functions $h^I(\phi)$ which are constrained:

$$C_{IJK} h^I h^J h^K = 1\,.$$

We introduce a function f and assume ($h_I \equiv C_{IJK} h^J h^K$)

$$h_I / f \equiv l_I + q_I \rho\,,$$

for some coordinate ρ. Let's define the central charge

$$\mathcal{Z}[\phi(\rho), q] \equiv h^I(\phi) q_I\,.$$

Then, using $h^I h_I = 1$ and $dh^I h_I = h^I dh_I = 0$

$$df^{-1} = d(h^I h_I / f) = h^I d(h_I / f)\,,$$

Then, using $h^I h_I = 1$ and $dh^I h_I = h^I dh_I = 0$

$$df^{-1} = d(h^I h_I / f) = h^I d(h_I / f)\,,$$

from which we get

$$\frac{df^{-1}}{d\rho} = \mathcal{Z}[\phi(\rho), q]\,.$$

Then, using $h^I h_I = 1$ and $dh^I h_I = h^I dh_I = 0$

$$df^{-1} = d(h^I h_I/f) = h^I d(h_I/f)\,,$$

from which we get

$$\frac{df^{-1}}{d\rho} = \mathcal{Z}[\phi(\rho), q]\,.$$

Using now the above properties plus $h^I{}_x h_{Iy} = g_{xy}$, where $h_{Iy} = -\sqrt{3}\partial_y h_I$ and $h^I{}_x = \sqrt{3}\partial_x h_I$

$$d\phi^x = h^{Ix} h_{Iy} d\phi^y = -\sqrt{3} h^{Ix} dh_I = -\sqrt{3} h^{Ix} d(f h_I/f) = -\sqrt{3} f h^{Ix} d(h_I/f)\,,$$

Then, using $h^I h_I = 1$ and $dh^I h_I = h^I dh_I = 0$

$$df^{-1} = d(h^I h_I/f) = h^I d(h_I/f)\,,$$

from which we get

$$\frac{df^{-1}}{d\rho} = \mathcal{Z}[\phi(\rho), q]\,.$$

Using now the above properties plus $h^I{}_x h_{Iy} = g_{xy}$, where $h_{Iy} = -\sqrt{3}\partial_y h_I$ and $h^I{}_x = \sqrt{3}\partial_x h_I$

$$d\phi^x = h^{Ix} h_{Iy} d\phi^y = -\sqrt{3} h^{Ix} dh_I = -\sqrt{3} h^{Ix} d(f h_I/f) = -\sqrt{3} f h^{Ix} d(h_I/f)\,,$$

from which we get

$$\frac{d\phi^x}{d\rho} = -f g^{xy}\partial_y \mathcal{Z}[\phi(\rho), q]\,.$$

The autonomous system of ordinary differential equations

$$\left\{\begin{array}{rcl} \frac{df^{-1}}{d\rho} & = & \mathcal{Z}[\phi(\rho), q]\,, \\ \\ \frac{d\phi^x}{d\rho} & = & -f g^{xy}\partial_y \mathcal{Z}[\phi(\rho), q]\,. \end{array}\right.$$

are the *black-hole attractor flow equations* of $N = 1, d = 5$ supergravity coupled to vector supermultiplets.

The autonomous system of ordinary differential equations

$$\begin{cases} \dfrac{df^{-1}}{d\rho} = \mathcal{Z}[\phi(\rho), q]\,, \\ \\ \dfrac{d\phi^x}{d\rho} = -f g^{xy}\partial_y \mathcal{Z}[\phi(\rho), q]\,. \end{cases}$$

are the *black-hole attractor flow equations* of $N = 1, d = 5$ supergravity coupled to vector supermultiplets.

The scalars will be attracted to the fixed points at which the r.h.s. vanishes:

$$\partial_y \mathcal{Z}[\phi, q]\Big|_{\phi=\phi_{\rm fix}} = 0\,, \qquad (\textit{Attractor equations}).$$

The autonomous system of ordinary differential equations

$$\begin{cases} \dfrac{df^{-1}}{d\rho} &= \mathcal{Z}[\phi(\rho), q]\,, \\ \\ \dfrac{d\phi^x}{d\rho} &= -f g^{xy}\partial_y \mathcal{Z}[\phi(\rho), q]\,. \end{cases}$$

are the *black-hole attractor flow equations* of $N = 1, d = 5$ supergravity coupled to vector supermultiplets.

The scalars will be attracted to the fixed points at which the r.h.s. vanishes:

$$\partial_y \mathcal{Z}[\phi, q]\Big|_{\phi=\phi_{\text{fix}}} = 0\,, \qquad (\textit{Attractor equations}).$$

ϕ_{fix} depends on the constants q_I and **not** on the constants l_I

$$\phi_{\text{fix}} = \phi_{\text{fix}}(q)$$

The autonomous system of ordinary differential equations

$$\begin{cases} \dfrac{df^{-1}}{d\rho} = \mathcal{Z}[\phi(\rho), q]\,, \\ \\ \dfrac{d\phi^x}{d\rho} = -f g^{xy}\partial_y \mathcal{Z}[\phi(\rho), q]\,. \end{cases}$$

are the *black-hole attractor flow equations* of $N = 1, d = 5$ supergravity coupled to vector supermultiplets.

The scalars will be attracted to the fixed points at which the r.h.s. vanishes:

$$\partial_y \mathcal{Z}[\phi, q]\Big|_{\phi=\phi_{\rm fix}} = 0\,, \qquad (\textit{Attractor equations}).$$

$\phi_{\rm fix}$ depends on the constants q_I and **not** on the constants l_I

$$\phi_{\rm fix} = \phi_{\rm fix}(q)$$

At the attractor point $\rho_{\rm attract}$ $\phi(\rho_{\rm attract}) = \phi_{\rm fix}$

$$\frac{df^{-1}}{d\rho}\Big|_{\rho=\rho_{\rm attract}} = \mathcal{Z}[\phi_{\rm fix}(q), q] \equiv \mathcal{Z}_{\rm fix}(q)\,.$$

Now for all $N \geq 2, d = 4$ supergravities

Now for all $N \geq 2, d = 4$ supergravities

Assume that, for some coordinate ρ $\mathcal{I} \equiv \mathcal{I}_0 + q\rho$.

Now for all $N \geq 2, d = 4$ supergravities

Assume that, for some coordinate ρ $\mathcal{I} \equiv \mathcal{I}_0 + q\rho$.

We define the central charges

$$\mathcal{Z}_{IJ}[\phi(\rho), q] \equiv \langle \mathcal{V}_{IJ} \mid q \rangle = p^{\Lambda} h_{\Lambda\, IJ} - q_{\Lambda} f^{\Lambda}{}_{IJ}\,,$$

$$\mathcal{Z}_{i}[\phi(\rho), q] \equiv \langle \mathcal{V}_{i} \mid q \rangle = p^{\Lambda} h_{\Lambda\, i} - q_{\Lambda} f^{\Lambda}{}_{i}\,.$$

Now for all $N \geq 2, d = 4$ supergravities

Assume that, for some coordinate ρ $\mathcal{I} \equiv \mathcal{I}_0 + q\rho$.

We define the central charges

$$\begin{aligned}
\mathcal{Z}_{IJ}[\phi(\rho), q] &\equiv \langle \mathcal{V}_{IJ} \mid q \rangle = p^\Lambda h_{\Lambda\, IJ} - q_\Lambda f^\Lambda{}_{IJ}\,, \\
\mathcal{Z}_{i}[\phi(\rho), q] &\equiv \langle \mathcal{V}_{i} \mid q \rangle = p^\Lambda h_{\Lambda\, i} - q_\Lambda f^\Lambda{}_{i}\,.
\end{aligned}$$

Then

$$\begin{aligned}
\mathfrak{D}\frac{M^{IJ}}{|M|^2} &= \mathfrak{D}\left(\frac{M^{KL}}{|M|^2}\tfrac{i}{2}\langle \mathcal{V}_{KL} \mid \mathcal{V}^{*\,IJ} \rangle\right) = \tfrac{i}{2}\mathfrak{D}\langle (\mathcal{R} + i\mathcal{I}) \mid \mathcal{V}^{*\,IJ} \rangle \\
&= \tfrac{i}{2}\langle d(\mathcal{R} + i\mathcal{I}) \mid \mathcal{V}^{*\,IJ} \rangle = \tfrac{i}{2}\langle d(\mathcal{R} - i\mathcal{I}) \mid \mathcal{V}^{*\,IJ} \rangle - \langle d\mathcal{I} \mid \mathcal{V}^{*\,IJ} \rangle \\
&= \tfrac{i}{2}\frac{M_{KL}}{|M|^2}\langle d\mathcal{V}^{*\,KL} \mid \mathcal{V}^{*\,IJ} \rangle - \langle q \mid \mathcal{V}^{*\,IJ} \rangle d\rho \\
&= \tfrac{1}{2}P^{*\,KLIJ}\frac{M_{KL}}{|M|^2} + \mathcal{Z}^{*\,IJ}[\phi(\rho), q]d\rho\,.
\end{aligned}$$

With the above identitiy we can compute

$$d|M|^{-2} = \frac{M_{IJ}}{|M|^2}\mathfrak{D}\frac{M^{IJ}}{|M|^2} + \frac{M^{IJ}}{|M|^2}\mathfrak{D}\frac{M_{IJ}}{|M|^2} = \frac{M_{IJ}\mathcal{Z}^{*\,IJ} + M^{IJ}\mathcal{Z}_{IJ}}{|M|^2}[\phi(\rho), q]d\rho\,,$$

With the above identitiy we can compute

$$d|M|^{-2} = \frac{M_{IJ}}{|M|^2}\mathfrak{D}\frac{M^{IJ}}{|M|^2} + \frac{M^{IJ}}{|M|^2}\mathfrak{D}\frac{M_{IJ}}{|M|^2} = \frac{M_{IJ}\mathcal{Z}^{*\,IJ} + M^{IJ}\mathcal{Z}_{IJ}}{|M|^2}[\phi(\rho), q]d\rho\,,$$

which leads to the flow equation (for all $N \geq 2$)

$$\frac{d}{d\rho}|M|^{-1} = \Re\mathrm{e}\left(\frac{M^{IJ}\mathcal{Z}_{IJ}}{|M|}\right)\,.$$

With the above identitiy we can compute

$$d|M|^{-2} = \frac{M_{IJ}}{|M|^2}\mathfrak{D}\frac{M^{IJ}}{|M|^2} + \frac{M^{IJ}}{|M|^2}\mathfrak{D}\frac{M_{IJ}}{|M|^2} = \frac{M_{IJ}\mathcal{Z}^{*\,IJ} + M^{IJ}\mathcal{Z}_{IJ}}{|M|^2}[\phi(\rho), q]d\rho\,,$$

which leads to the flow equation (for all $N \geq 2$)

$$\frac{d}{d\rho}|M|^{-1} = \Re\mathrm{e}\left(\frac{M^{IJ}\mathcal{Z}_{IJ}}{|M|}\right)\,.$$

We can also compute

$$0 = M^{[IJ}\mathfrak{D}\frac{M^{KL]}}{|M|^2} = M^{[IJ}\mathcal{Z}^{*\,KL]}[\phi(\rho), q]d\rho + \tfrac{1}{2}P^{*\,MN[IJ}\mathcal{J}^{K}{}_{M}\mathcal{J}^{L]}{}_{N}\,,$$

With the above identitiy we can compute

$$d|M|^{-2} = \frac{M_{IJ}}{|M|^2}\mathfrak{D}\frac{M^{IJ}}{|M|^2} + \frac{M^{IJ}}{|M|^2}\mathfrak{D}\frac{M_{IJ}}{|M|^2} = \frac{M_{IJ}\mathcal{Z}^{*\,IJ} + M^{IJ}\mathcal{Z}_{IJ}}{|M|^2}[\phi(\rho), q]d\rho\,,$$

which leads to the flow equation (for all $N \geq 2$)

$$\frac{d}{d\rho}|M|^{-1} = \Re\mathrm{e}\left(\frac{M^{IJ}\mathcal{Z}_{IJ}}{|M|}\right)\,.$$

We can also compute

$$0 = M^{[IJ}\mathfrak{D}\frac{M^{KL]}}{|M|^2} = M^{[IJ}\mathcal{Z}^{*\,KL]}[\phi(\rho), q]d\rho + \tfrac{1}{2}P^{*\,MN[IJ}\mathcal{J}^{K}{}_{M}\mathcal{J}^{L]}{}_{N}\,,$$

which leads to the flow equation ($N \geq 4$)

$$P^{*\,MN[IJ}\mathcal{J}^{K}{}_{M}\mathcal{J}^{L]}{}_{N} = -M^{[IJ}\mathcal{Z}^{*\,KL]}[\phi(\rho), q]d\rho\,.$$

The third flow equation ($N = 2, 3, 4, 6$) follows from

$$\begin{aligned} \tfrac{1}{2}\frac{M^{IJ}}{|M|^2}P_{iIJ} &= -\tfrac{i}{2}\frac{M^{IJ}}{|M|^2}\langle\, d\mathcal{V}_{IJ} \mid \mathcal{V}_i\,\rangle = -\tfrac{i}{2}\langle\, d(\mathcal{R}+i\mathcal{I}) \mid \mathcal{V}_i\,\rangle \\ &= \langle\, d\mathcal{I} \mid \mathcal{V}_i\,\rangle - \tfrac{i}{2}\langle\, d(\mathcal{R}-i\mathcal{I}) \mid \mathcal{V}_i\,\rangle \\ &= -\mathcal{Z}_i[\phi(\rho), q]d\rho\,, \end{aligned}$$

The third flow equation ($N = 2, 3, 4, 6$) follows from

$$\begin{aligned}
\tfrac{1}{2}\frac{M^{IJ}}{|M|^2}P_{iIJ} &= -\tfrac{i}{2}\frac{M^{IJ}}{|M|^2}\langle\, d\mathcal{V}_{IJ} \mid \mathcal{V}_i\,\rangle = -\tfrac{i}{2}\langle\, d(\mathcal{R}+i\mathcal{I}) \mid \mathcal{V}_i\,\rangle \\
&= \langle\, d\mathcal{I} \mid \mathcal{V}_i\,\rangle - \tfrac{i}{2}\langle\, d(\mathcal{R}-i\mathcal{I}) \mid \mathcal{V}_i\,\rangle \\
&= -\mathcal{Z}_i[\phi(\rho), q]d\rho\,,
\end{aligned}$$

and takes the final form

$$P_{i\,KL}\mathcal{J}^K{}_I\mathcal{J}^L{}_J = -2M_{IJ}\mathcal{Z}_i[\phi(\rho), q]d\rho\,.$$

The third flow equation ($N = 2, 3, 4, 6$) follows from

$$\begin{aligned} \tfrac{1}{2}\frac{M^{IJ}}{|M|^2} P_{iIJ} &= -\tfrac{i}{2}\frac{M^{IJ}}{|M|^2}\langle\, d\mathcal{V}_{IJ} \mid \mathcal{V}_i \,\rangle = -\tfrac{i}{2}\langle\, d(\mathcal{R} + i\mathcal{I}) \mid \mathcal{V}_i \,\rangle \\ &= \langle\, d\mathcal{I} \mid \mathcal{V}_i \,\rangle - \tfrac{i}{2}\langle\, d(\mathcal{R} - i\mathcal{I}) \mid \mathcal{V}_i \,\rangle \\ &= -\mathcal{Z}_i[\phi(\rho), q] d\rho \,, \end{aligned}$$

and takes the final form

$$P_{i\,KL}\mathcal{J}^K{}_I\mathcal{J}^L{}_J = -2M_{IJ}\mathcal{Z}_i[\phi(\rho), q] d\rho \,.$$

These flow equations lead to the generic N *attractor equations* (work in progress).

13 – Final comments

13 – Final comments

★ We have found the general form of all the timelike supersymmetric solutions of all $d = 4$ supergravities .

13 – Final comments

- ★ We have found the general form of all the timelike supersymmetric solutions of all $d = 4$ supergravities .

- ★ We have proven the relation between the timelike supersymmetric solutions of all $d = 4$ supergravities and those of the $N = 2$ theories (for black holes conjectured by Ferrara, Gimon & Kallosh (2006) and proven by Bossard (2010)).

13 – Final comments

- We have found the general form of all the timelike supersymmetric solutions of all $d = 4$ supergravities .

- We have proven the relation between the timelike supersymmetric solutions of all $d = 4$ supergravities and those of the $N = 2$ theories (for black holes conjectured by Ferrara, Gimon & Kallosh (2006) and proven by Bossard (2010)).

- We have shown how the would-be scalars in vector multiplets and hypermultiplets can be distinguished and we have shown that the *attractor mechanism* only acts on the former.

13 – Final comments

★ We have found the general form of all the timelike supersymmetric solutions of all $d = 4$ supergravities .

★ We have proven the relation between the timelike supersymmetric solutions of all $d = 4$ supergravities and those of the $N = 2$ theories (for black holes conjectured by Ferrara, Gimon & Kallosh (2006) and proven by Bossard (2010)).

★ We have shown how the would-be scalars in vector multiplets and hypermultiplets can be distinguished and we have shown that the *attractor mechanism* only acts on the former.

★ We have given "1-line" derivations of the attactor flow equations.

13 – Final comments

- ★ We have found the general form of all the timelike supersymmetric solutions of all $d = 4$ supergravities .

- ★ We have proven the relation between the timelike supersymmetric solutions of all $d = 4$ supergravities and those of the $N = 2$ theories (for black holes conjectured by Ferrara, Gimon & Kallosh (2006) and proven by Bossard (2010)).

- ★ We have shown how the would-be scalars in vector multiplets and hypermultiplets can be distinguished and we have shown that the *attractor mechanism* only acts on the former.

- ★ We have given "1-line" derivations of the attactor flow equations.

- ★ Much work remains to be done in order to make explicit the construction of the solutions. In particular one has to find general parametrizations of the matrices M^{IJ} and $\mathcal{J}^{I}{}_{J}$, solve the *stabilization equations*, impose the covariant constancy of $\mathcal{J}$ etc. (Meessen & O., work in progress).