All the d=4 timelike supersymmetric solutions

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Work done in collaboration with *P. Meessen* (University of Oviedo) and *S. Vaulà* (IFT UAM/CSIC, Madrid)

Plan of the Talk:

1 – Introduction: the search for all 4-d susy solutions

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- **2006**: Bellorín, Meessen & O. ($N = 1$ $d = 5$ with vector multiplets); Meessen & O. ($N = 2$ $d = 4$ with vector multiplets); Hübscher, Meessen & O. ($N = 2$ $d = 4$ with vector multiplets and hypermultiplets).

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> For $N > 2$ there are too many spinor bilinears and we do not know how to extract the (**not** spacetime-geometric) information they must surely contain.

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☞ Black-hole attractors **1996**: Ferrara, Kallosh & Strominger.

This mechanism can be used as a powerful tool to find partial information about extremal (supersymmetric and non-supersymmetric) black holes.

These methods give complementary information.

However, in our opinion, the spinor-bilinear method would give the most if we could solve its problems for $N > 2$.

In this talk we are going to show how to solve those problems and determine the form of **all** the timelike supersymmetric solutions of all $d = 4$ supergravities using the **spinor-bilinear method**.

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$$\left\{e^{a}{}_{\mu}, \psi_{I\,\mu}, A^{IJ}{}_{\mu}\right\}\,, \quad I, J, \cdots = 1, 2\,, \quad \Rightarrow A^{IJ}{}_{\mu} = A^{0}{}_{\mu}\varepsilon^{IJ}\,.$$

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This is a extremely redundant (but useful) description of the scalars .

The supersymmetry transformations of the fermions are

$$\begin{aligned}
\delta_\epsilon \psi_{I\,\mu} &= \mathfrak{D}_\mu \epsilon_I + \varepsilon_{IJ}\, T^+{}_{\mu\nu}\gamma^\nu\, \epsilon^J\,, \\
\delta_\epsilon \lambda^{iI} &= i\, \not{\partial} Z^i \epsilon^I \,+\, \varepsilon^{IJ}\, \not{G}^{i\,+}\, \epsilon_J\,. \\
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where the graviphoton and matter vector field strengths are

$$T^+ = \langle\, \mathcal{V} \mid \mathcal{F}^+ \,\rangle\,, \qquad G^{i\,+} = \tfrac{i}{2}\mathcal{G}^{ij^*}\langle\, \mathcal{D}_{j^*}\mathcal{V}^* \mid \mathcal{F}^+ \,\rangle\,, \qquad \mathcal{F}^+ \equiv \begin{pmatrix} F^{\Lambda\,+} \\ \mathcal{N}^*_{\Lambda\Sigma}F^{\Sigma\,+} \end{pmatrix}\,,$$

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$\mathfrak{D}$ is the Lorentz-, Kähler- and $SU(2)$- covariant derivative (Kähler + $SU(2) = U(2)$)

$$\mathfrak{D}_\mu \epsilon_I = \left(\partial_\mu + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} \,+\, \tfrac{i}{2}\,\mathcal{Q}_\mu\right)\epsilon_I \,+\, \mathsf{A}_{\mu\,I}{}^J\,\epsilon_J\,,$$

and where $\mathsf{U}^{\alpha I}{}_u(q)$ is the *Quadbein*.

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and where $\mathsf{U}^{\alpha I}{}_u(q)$ is the *Quadbein*. The action for the bosonic fields is

$$\begin{aligned}
S\ \ &=\ \ \int d^4x\sqrt{|g|}\,\big[R + 2\mathcal{G}_{ij^*}\partial_\mu Z^i \partial^\mu Z^{*j^*} + 2\mathsf{H}_{uv}\partial_\mu q^u \partial^\mu q^v \\
&\qquad +2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}F^\Sigma{}_{\mu\nu} - 2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}\star F^\Sigma{}_{\mu\nu}\big]\,.
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The goal is to find **all** the bosonic field configurations $\{e^a{}_\mu, A^\Lambda{}_\mu, Z^i, q^u\}$ such that the above KSEs admit at least one solution ϵ^I.

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5. Impose the independent equations of motion on the supersymmetric configurations we just identified.

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The 4-d Fierz identities imply that $V_a \equiv V^I{}_{I\,a}$ is always non-spacelike:

$$V^2 = -V^I{}_J \cdot V^J{}_I = 2M^{IJ} M_{IJ} = 4|X|^2 \geq 0\,.$$

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With them one can construct a tetrad

$$V^a{}_\mu \equiv \tfrac{1}{\sqrt{2}} V^I{}_{J\,\mu} (\sigma^a)^J{}_I\,, \qquad V^I{}_{J\,\mu} = \tfrac{1}{\sqrt{2}} V^a{}_\mu (\sigma^a)^I{}_J\,,$$

with $\sigma^0 = 1$ and σ^m the 2×2 Pauli matrices as an orthonormal tetrad in which $V^0 = \sqrt{2} V$ is timelike and the V^ms are spacelike.

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1. A complex antisymmetric matrix of scalars $M_{IJ} \equiv \bar{\epsilon}_I \epsilon_J = X \varepsilon_{IJ}$.
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2. A Hermitean matrix of vectors $V^I{}_{J\,a} \equiv i\bar{\epsilon}^I \gamma_a \epsilon_J$.

The 4-d Fierz identities imply that $V_a \equiv V^I{}_{I\,a}$ is always non-spacelike:

$$V^2 = -V^I{}_J \cdot V^J{}_I = 2M^{IJ} M_{IJ} = 4|X|^2 \geq 0\,.$$

We only consider the timelike case $X \neq 0$ in which all $V^I{}_{J\,a}$ are independent.

With them one can construct a tetrad

$$V^a{}_\mu \equiv \tfrac{1}{\sqrt{2}} V^I{}_{J\,\mu} (\sigma^a)^J{}_I\,, \qquad V^I{}_{J\,\mu} = \tfrac{1}{\sqrt{2}} V^a{}_\mu (\sigma^a)^I{}_J\,,$$

with $\sigma^0 = 1$ and σ^m the 2×2 Pauli matrices as an orthonormal tetrad in which $V^0 = \sqrt{2} V$ is timelike and the V^ms are spacelike. (**This will not work for $N > 2$!**)

5 – The $N = 2$ Killing Spinor Identities (KSI)s

If we assume that a given bosonic field configuration admits a Killing spinor ϵ_I, then we find that the (*off-shell*) "equations of motion" $\{\mathcal{E}^{\mu\nu}, \mathcal{E}^{\mu}, \mathcal{E}^{i}, \mathcal{E}_{u}\}$ satisfy the KSIs:

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6. $\mathcal{E}_{i^*} = 2\left(\frac{X}{X^*}\right)^{1/2} \langle\, \mathcal{E}^0 \mid \mathcal{D}_{i^*}\mathcal{V}^*\,\rangle$, ($\Rightarrow$ attractor mechanism)

The only independent equations of motion that have to be imposed on $N = 2$, $d = 4$ supersymmetric configurations are

$$\mathcal{E}^0 = 0\,.$$

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4. The scalars Z^i are given by the quotients

$$Z^i = \frac{\mathcal{V}^i/X}{\mathcal{V}^0/X} = \frac{\mathcal{R}^i + i\mathcal{I}^i}{\mathcal{R}^0 + i\mathcal{I}^0}\,.$$

5. The hyperscalars $q^u(x)$ are the mappings satisfying

$$\mathsf{U}^{\alpha J}{}_{m}\,(\sigma^{m})_{J}{}^{I} \;=\; 0\,, \qquad\qquad \mathsf{U}^{\alpha J}{}_{n} \;\equiv\; V_{n}{}^{\underline{m}}\partial_{\underline{m}}q^{u}\;\mathsf{U}^{\alpha J}{}_{u}\,.$$

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$\gamma_{\underline{mn}}$ is determined indirectly from the hyperscalars : its spin connection ϖ^{mn} in the basis $\{V^m\}$ is related to the pullback of the $SU(2)$ connection of the hyper-Kähler manifold $\mathsf{A}^I{}_{J\mu} = \frac{1}{\sqrt{2}} \mathsf{A}^m{}_u (\sigma^m)^I{}_J \partial_\mu q^u$, by

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7. The vector field strengths are

$$\mathcal{F} = -\tfrac{1}{2}d(\mathcal{R}\hat{V}) - \tfrac{1}{2}\star(\hat{V}\wedge d\mathcal{I})\,, \qquad \hat{V} = 2\sqrt{2}|X|^2(dt+\omega)\,.$$

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All 4-d supergravity multiplets can be written in the form

$$\left\{e^{a}{}_{\mu}, \psi_{I\,\mu}, A^{IJ}{}_{\mu}, \chi_{IJK}, P_{IJKL\,\mu}, \chi^{IJKLM}\right\}, \quad I, J, \cdots = 1, \cdots, N,$$

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The price to pay for using this representation is that all the fields that can be related by $SU(N)$ duality relations, are:

- $N = 4$: $P^{*\,i\,IJ} = \frac{1}{2}\varepsilon^{IJKL}P_{i\,KL}$, and $\lambda_{iI} = \frac{1}{3!}\varepsilon_{IJKL}\lambda_i{}^{IJK}$.
- $N = 6$: $P^{*\,IJ} = \frac{1}{4!}\varepsilon^{IJK_1\cdots K_4}P_{K_1\cdots K_4}$, $\chi_{IJK} = \frac{1}{3!}\varepsilon_{IJKLMN}\lambda^{IJK}$, and $\chi^{I_1\cdots I_5} = \varepsilon^{I_1\cdots I_5 J}\lambda_J$.
- $N = 8$: $P^{*\,I_1\cdots I_4} = \frac{1}{4!}\varepsilon^{I_1\cdots I_4 J_1\cdots J_4}P_{J_1\cdots J_4}$, and $\chi_{I_1I_2I_3} = \frac{1}{5!}\varepsilon_{I_1I_2I_3J_1\cdots J_5}\chi^{J_1\cdots J_5}$. These constraints must be taken into account in the action.

The scalars are encoded into the $2\bar{n}$-dimensional ($\bar{n} \equiv n + \frac{N(N-1)}{2}$) symplectic vectors

$$\mathcal{V}_{IJ} = \begin{pmatrix} f^{\Lambda}{}_{IJ} \\ h_{\Lambda\, IJ} \end{pmatrix}, \qquad \text{and} \qquad \mathcal{V}_{i} = \begin{pmatrix} f^{\Lambda}{}_{i} \\ h_{\Lambda\, i} \end{pmatrix}, \qquad \Lambda = 1, \cdots, \bar{n},$$

normalized

$$\langle \mathcal{V}_{IJ} \mid \mathcal{V}^{*\,KL} \rangle = -2i\delta^{KL}{}_{IJ}, \qquad \langle \mathcal{V}_{i} \mid \mathcal{V}^{*\,j} \rangle = -i\delta_{i}{}^{j}.$$

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They can be combined into the $Usp(\bar{n}, \bar{n})$ matrix

$$U \equiv \tfrac{1}{\sqrt{2}} \begin{pmatrix} f + ih & f^* + ih^* \\ f - ih & f^* - ih^* \end{pmatrix}.$$

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The graviphotons $A^{IJ}{}_{\mu}$ do not appear directly, only through the “dressed” vectors

$$A^{\Lambda}{}_{\mu} \equiv \tfrac{1}{2} f^{\Lambda}{}_{IJ} A^{IJ}{}_{\mu} + f^{\Lambda}{}_{i} A^{i}{}_{\mu}\,.$$

The supersymmetry transformations of the fermioninc fields are

$$\begin{aligned}
\delta_\epsilon \psi_{I\mu} &= \mathfrak{D}_\mu \epsilon_I + T_{IJ}{}^+{}_{\mu\nu}\gamma^\nu \epsilon^J\,, \\
\delta_\epsilon \chi_{IJK} &= -\tfrac{3i}{2}\, \not{T}_{[IJ}{}^+ \epsilon_{K]} + i\, \not{P}_{IJKL}\epsilon^L\,, \\
\delta_\epsilon \lambda_{iI} &= -\tfrac{i}{2}\, \not{T}_i{}^+ \epsilon_I + i\, \not{P}_{iIJ}\epsilon^J\,, \\
\delta_\epsilon \chi_{IJKLM} &= -5i\, \not{P}_{[IJKL}\epsilon_{M]} + \tfrac{i}{2}\varepsilon_{IJKLMN}\, \not{T}^- \epsilon^N + \tfrac{i}{4}\varepsilon_{IJKLMNOP}\, \not{T}^{NO-}\epsilon^P\,, \\
\delta_\epsilon \lambda_{iIJK} &= -3i\, \not{P}_{i[IJ}\epsilon_{K]} + \tfrac{i}{2}\varepsilon_{IJKL}\, \not{T}_i{}^- \epsilon^L + \tfrac{i}{4}\varepsilon_{IJKLMN}\, \not{T}^{LM-}\epsilon_N\,,
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where the graviphoton and matter vector field strengths are

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T_{IJ}{}^{+} = \langle\, \mathcal{V}_{IJ} \mid \mathcal{F}^{+}\,\rangle\,, \quad T_{i}{}^{+} = \langle\, \mathcal{V}_{i} \mid \mathcal{F}^{+}\,\rangle\,, \quad \mathcal{F}_{\Lambda}{}^{+} = \mathcal{N}^{*}_{\Lambda\Sigma} F^{\Sigma\,+}\,,
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and where

$$\mathfrak{D}_\mu \epsilon_I \equiv \nabla_\mu \epsilon_I - \epsilon_J \Omega_\mu{}^J{}_I\,,$$

and $\Omega_\mu{}^J{}_I$ is the pullback of the connection of the scalar manifold ($\subset U(N)$).

The action for the bosonic fields is

$$S = \int d^4x \sqrt{|g|}\,\left[R + 2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}F^{\Sigma}{}_{\mu\nu} - 2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}\star F^{\Sigma}{}_{\mu\nu}\right.$$

$$\left. + \tfrac{2}{4!}\alpha_1 P^{*\,IJKL}{}_{\mu}P_{IJKL}{}^{\mu} + \alpha_2 P^{*\,iIJ}{}_{\mu}P_{iIJ}{}^{\mu}\right]\,,$$

where

$$\mathcal{N} = hf^{-1} = \mathcal{N}^{T}\,, \qquad h_{\Lambda} = \mathcal{N}_{\Lambda\Sigma}f^{\Sigma}\,. \qquad \mathfrak{D}h_{\Lambda} = \mathcal{N}^{*}_{\Lambda\Sigma}\mathfrak{D}f^{\Lambda}\,.$$

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For $N=2$ **:** $\mathcal{E}^{iIJ} = \mathfrak{D}^{\mu}P^{*\,iIJ}{}_{\mu} + 2T^{i\,-}{}_{\mu\nu}T^{IJ\,-\,\mu\nu} + P^{*\,iIJ\,A}P^{*\,jk}{}_{A}T_{j}{}^{+}{}_{\mu\nu}T_{k}{}^{+\,\mu\nu}\,.$

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$$+\tfrac{2}{4!}\alpha_1 P^{*\,IJKL}{}_{\mu}P_{IJKL}{}^{\mu} + \alpha_2 P^{*\,iIJ}{}_{\mu}P_{iIJ}{}^{\mu}\big]\,,$$

where

$$\mathcal{N} = hf^{-1} = \mathcal{N}^T\,, \qquad h_\Lambda = \mathcal{N}_{\Lambda\Sigma}f^{\Sigma}\,. \qquad \mathfrak{D}h_\Lambda = \mathcal{N}^*_{\Lambda\Sigma}\mathfrak{D}f^{\Lambda}\,.$$

The N-specific constraints must be taken into account to find the e.o.m.:

For $N=2$ **:** $\mathcal{E}^{iIJ} = \mathfrak{D}^{\mu}P^{*\,iIJ}{}_{\mu} + 2T^{i\,-}{}_{\mu\nu}T^{IJ\,-\,\mu\nu} + P^{*\,iIJ\,A}P^{*\,jk}{}_{A}T_j{}^{+}{}_{\mu\nu}T_k{}^{+\,\mu\nu}$.

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For $N = 3$ **:** $\mathcal{E}^{iIJ} = \mathfrak{D}^{\mu}P^{*\,iIJ}{}_{\mu} + 2T^{i\,-}{}_{\mu\nu}T^{IJ\,-\,\mu\nu}.$

For $N = 4$ **:**

$$\left\{\begin{array}{rcl} \mathcal{E}^{IJKL} & = & \mathfrak{D}^{\mu}P^{*\,IJKL}{}_{\mu} + 6T^{[IJ|-}{}_{\mu\nu}T^{|KL]-\,\mu\nu} \\ & & +P^{*\,IJKL\,A}P^{*\,ij}{}_{A}T_{i}{}^{+}{}_{\mu\nu}T_{j}{}^{+\,\mu\nu}\,, \\ \mathcal{E}^{iIJ} & = & \mathfrak{D}^{\mu}P^{*\,iIJ}{}_{\mu} + T^{i\,-}{}_{\mu\nu}T^{IJ\,-\,\mu\nu} + \tfrac{1}{2}\varepsilon^{IJKL}T_{i}{}^{+}{}_{\mu\nu}T_{KL}{}^{+\,\mu\nu}\,. \end{array}\right.$$

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For $N=5$ **:** $\mathcal{E}^{IJKL} = \mathfrak{D}^{\mu}P^{*\,IJKL}{}_{\mu} + 6T^{[IJ|-}{}_{\mu\nu}T^{|KL]-\,\mu\nu}$. etc.

8 – The all-N Killing Spinor Equations (KSEs)

For all values of N the independent KSEs take the form

$$
\begin{aligned}
\mathfrak{D}_\mu \epsilon_I + T_{IJ}{}^{+}{}_{\mu\nu}\gamma^\nu \epsilon^J &= 0\,, \\
\not{P}_{IJKL}\epsilon^L - \tfrac{3}{2}\not{T}_{[IJ}{}^{+}\epsilon_{K]} &= 0\,, \\
\not{P}_{i\,IJ}\epsilon^J - \tfrac{1}{2}\not{T}_i{}^{+}\epsilon_I &= 0\,, \\
\not{P}_{[IJKL}\epsilon_{M]} &= 0\,, \\
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Again, our goal is to find **all** the bosonic field configurations $\{e^a{}_\mu, A^\Lambda{}_\mu, P_{IJKL\,\mu}, P_{i\,IJ\,\mu}\}$ such that the above KSEs admit at least one solution ϵ^I.

9 – The all-N spinor-bilinears algebra

The independent bilinears that we can construct with one $U(N)$ vector of Weyl spinors ϵ_I are:

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   We only consider the timelike case.
3. We can choose a tetrad $\{e^a{}_\mu\}$ such that $e^0{}_\mu \equiv \frac{1}{\sqrt{2}} |M|^{-1} V_\mu$. Then, **defining** $V^m{}_\mu \equiv |M| e^m{}_\mu$ we can decompose

$$V^I{}_{J\,\mu} = \tfrac{1}{2} \mathcal{J}^I{}_J V_\mu + \tfrac{1}{\sqrt{2}} (\sigma^m)^I{}_J V^m{}_\mu \,,$$

where $\mathcal{J}^I{}_J = 2M^{IK} M_{JK} |M|^{-2}$ is a rank 2 projector (Tod):

$$\mathcal{J}^2 = \mathcal{J} \,, \qquad \mathcal{J}^I{}_I = +2 \,, \qquad \mathcal{J}^I{}_J \epsilon^J = \epsilon^I \,.$$

The main properties satisfied by the three σ^m matrices are:

$$
\begin{aligned}
\sigma^m \sigma^n &= \delta^{mn} \mathcal{J} + i \varepsilon^{mnp} \sigma^p \,, \\
\mathcal{J} \sigma^m &= \sigma^m \mathcal{J} = \sigma^m \,, \\
(\sigma^m)^I{}_I &= 0 \,, \\
\mathcal{J}^K{}_J \mathcal{J}^L{}_I &= \tfrac{1}{2} \mathcal{J}^K{}_I \mathcal{J}^L{}_J + \tfrac{1}{2} (\sigma^m)^K{}_I (\sigma^m)^L{}_J \,, \\
M_{K[I} (\sigma^m)^K{}_{J]} &= 0 \,, \\
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$\{\mathcal{J}, \sigma^1, \sigma^2, \sigma^3\}$ is an x-dependent basis of a $\mathfrak{u}(2)$ subalgebra of $\mathfrak{u}(N)$ in the 2-dimensional eigenspace of $\mathcal{J}$ of eigenvalue $+1$ and provide a basis in the space of Hermitean matrices A satisfying $\mathcal{J} A \mathcal{J} = A$

10 – The all-N Killing Spinor Identities (KSIs)

If we assume that a given bosonic field configuration admits a Killing spinor ϵ_I, then we find that the (*off-shell*) "equations of motion" $\{\mathcal{E}^{\mu\nu}, \mathcal{E}^{\mu}, \mathcal{E}^{IJKL}, \mathcal{E}^{i\,IJ}\}$ satisfy the KSIs ($\tilde{\mathcal{J}}^{I}{}_{J} \equiv \delta^{I}{}_{J} - \mathcal{J}^{I}{}_{J}$):

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$$\left\{ \begin{array}{rcl} \mathcal{E}^{MNPQ}\mathcal{J}^{[I}{}_{M}\tilde{\mathcal{J}}^{J}{}_{N}\tilde{\mathcal{J}}^{K}{}_{P}\tilde{\mathcal{J}}^{L]}{}_{Q} & = & 0\,, \\ \mathcal{E}^{i\,MN}\mathcal{J}^{[I}{}_{M}\tilde{\mathcal{J}}^{J]}{}_{N} & = & 0\,, \end{array} \right.$$
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4. $\mathcal{E}^{00} = -2\sqrt{2}\langle\, \mathcal{E}^0 \mid \Re\mathrm{e}\left(\mathcal{V}_{IJ}\frac{M^{IJ}}{|M|}\right)\rangle$, (Bogomol'nyi bound)

5. $\langle \mathcal{E}^0 \mid \Im\mathrm{m}\left(\mathcal{V}_{IJ}\frac{M^{IJ}}{|M|}\right)\rangle$, ($\Rightarrow$ no NUT charge).

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$$N = 3 : \; \mathcal{E}^{i\,IJ} = -2\sqrt{2}\frac{M^{IJ}}{|M|}\langle\, \mathcal{E}^0 \mid \mathcal{V}^{*\,i}\,\rangle\,,$$

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$$N=4: \begin{cases} \mathcal{E}^{IJKL} = -2\sqrt{2}\frac{M^{[IJ|}}{|M|}\langle \mathcal{E}^0 \mid \mathcal{V}^{*\,|KL]}\rangle\,, \\ \mathcal{E}_{iIJ} = -2\sqrt{2}\left\{\frac{M_{IJ}}{|M|}\langle \mathcal{E}^0 \mid \mathcal{V}_i\rangle + \tfrac{1}{2}\varepsilon_{IJKL}\frac{M^{KL}}{|M|}\langle \mathcal{E}^0 \mid \mathcal{V}^{*\,i}\rangle\right\}\,, \end{cases}$$

etc.

The only independent equations of motion that have to be imposed on **any** $d = 4$ supersymmetric configuration are

$$\mathcal{E}^0 = 0\,.$$

11 – The all-N supersymmetric solutions

The construction of any timelike supersymmetric solution proceeds as follows:

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   We also have to impose the constraint

   $$\mathcal{J} d\sigma^m \mathcal{J} = 0\,.$$

Once the $U(2)$ subgroup has been chosen, we can split the Vielbeins $P_{IJKL\,\mu}$ and $P_{i\,IJ\,\mu}$, into associated to the would-be vector multiplets in the $N=2$ truncation

$$P_{IJKL}\,\mathcal{J}^{I}{}_{[M}\mathcal{J}^{J}{}_{N}\tilde{\mathcal{J}}^{K}{}_{P}\tilde{\mathcal{J}}^{L}{}_{Q]}\,, \quad \text{and} \quad P_{i\,IJ}\,\mathcal{J}^{I}{}_{[K}\mathcal{J}^{J}{}_{L]}\,,$$

which are driven by the *attractor mechanism* (*i.e.* they are determined by the electric and magnetic charges) and those associated to the hypermultiplets

$$P_{IJKL}\,\mathcal{J}^{I}{}_{[M}\tilde{\mathcal{J}}^{J}{}_{N}\tilde{\mathcal{J}}^{K}{}_{P}\tilde{\mathcal{J}}^{L}{}_{Q]}\,, \quad \text{and} \quad P_{i\,IJ}\,\mathcal{J}^{I}{}_{[K}\tilde{\mathcal{J}}^{J}{}_{L]}\,.$$

which are not.

In hyper-less solutions (*e.g.* black holes) the σ^{m}s matrices are not needed at all.

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3. $\mathcal{R}$ is to be be found from $\mathcal{I}$ by solving the generalized *stabilization equations*.

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where

$$\begin{aligned} |M|^{-2} &= (M^{IJ}M_{IJ})^{-2} = \langle\,\mathcal{R}\mid\mathcal{I}\,\rangle\,, \\ (d\omega)_{mn} &= 2\epsilon_{mnp}\langle\,\mathcal{I}\mid\partial^p\mathcal{I}\,\rangle\,. \end{aligned}$$

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6. The scalars in the vector multiplets in the associated $N=2$ truncation

$$P_{IJKL}\,\mathcal{J}^I{}_{[M}\mathcal{J}^J{}_N\tilde{\mathcal{J}}^K{}_P\tilde{\mathcal{J}}^L{}_{Q]}, \quad \text{and} \quad P_{i\,IJ}\,\mathcal{J}^I{}_{[K}\mathcal{J}^J{}_{L]},$$

can be found from $\mathcal{R}$ and $\mathcal{I}$, while those in the hypers must be found independently by solving

$$P_{IJKL\,m}\,\mathcal{J}^I{}_{[M}\tilde{\mathcal{J}}^J{}_N\tilde{\mathcal{J}}^K{}_P\tilde{\mathcal{J}}^L{}_{Q]}(\sigma^m)^Q{}_R = 0,$$

$$P_{i\,IJ\,m}\,\mathcal{J}^I{}_{[K}\tilde{\mathcal{J}}^J{}_{L]}(\sigma^m)^L{}_M = 0,$$

which solve their equations of motion according to the *Killing Spinor Identities*.

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★ Much work remains to be done in order to make explicit the construction of the solutions. In particular one has to find general parametrizations of the matrices M^{IJ} and $\mathcal{J}^{I}{}_{J}$, solve the *stabilization equations*, impose the covariant constancy of $\mathcal{J}$ etc. (Meessen & O., work in progress).

A simple derivation of the attractor flow eqs. in $N=1, d=5$ supergravity

Assume

$$h_I/f \equiv l_I + q_I\rho\,,$$

and define the central charge

$$\mathcal{Z}[\phi(\rho),q] \equiv h^I(\phi)q_I\,.$$

Using $h^I h_I = 1$ and $h^I dh_I = 0$

$$df^{-1} = d(h^I h_I/f) = h^I d(h_I/f)\,,$$

from which we get

$$\frac{df^{-1}}{d\rho} = \mathcal{Z}[\phi(\rho),q]\,.$$

Using now the above properties plus $h^I{}_x h_{Iy} = g_{xy}$, where $h_{Iy} = -\sqrt{3}\partial_y h_I$ and $h^I{}_x = \sqrt{3}\partial_x h_I$

$$d\phi^x = h^{Ix}h_{Iy}d\phi^y = -\sqrt{3}h^{Ix}dh_I = -\sqrt{3}h^{Ix}d(fh_I/f) = -\sqrt{3}fh^{Ix}d(h_I/f)\,,$$

from which we get

$$\frac{d\phi^x}{d\rho} = -fg^{xy}\partial_y\mathcal{Z}[\phi(\rho),q]\,.$$