Some new results on extremal and non-extremal black holes

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published in `arXiv:1105.3311` , `arXiv:1107.5454`, `arXiv:1112.3332`, `arXiv:1203.0260`, `arXiv:1204.0507`, `arXiv:1204.5910`, `arXiv:1206.3190` and work to appear.

Talk given at Workshop INFN-Spain 2012, Naples November 13th, 2012

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In this talk I will present a general ansatz and a general formalism to construct non-extremal black-hole and black-brane solutions. Then we can take their extremal non-supersymmetric limits.
I will review a complete explicit example.

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We will prove the ansatz constructing a new formalism (*H-FGK formalism*) which simplifies the construction of solutions and the study of general properties of families of black holes.

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We start by reviewing the FGK formalism

for black holes and black branes

in d dimensions.

2 – FGK formalism for black p-branes in d dimensions

Consider the generic d-dimensional spacetime action describing scalars ϕ^i and $(p+1)$-form potentials $A^{\Lambda}_{(p+1)}$ coupled to gravity:

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\begin{aligned}
I &= \int d^dx\sqrt{|g|}\,\{R + \mathcal{G}_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j \\
&\quad +4\tfrac{(-1)^p}{(p+2)!}\left[I_{\Lambda\Sigma}(\phi)F^{\Lambda}_{(p+2)}\cdot F^{\Sigma}_{(p+2)} + \xi^2 R_{\Lambda\Sigma}(\phi)F^{\Lambda}_{(p+2)}\star F^{\Sigma}_{(p+2)}\right]\Big\}\,,
\end{aligned}
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where the last term occurs only when $p=\tilde{p}=(d-4)/2$ and

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R_{\Lambda\Sigma}(\phi) = -\xi^2 R_{\Sigma\Lambda}(\phi)\,, \qquad \xi^2 = (-1)^{\frac{d}{2}+1} = (-1)^{p+1}\,.
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We want to find solutions describing single, static, charged, regular, black p-branes with flat worldvolume in the directions $\vec{y}_{(p)}=(y_1,\cdots,y_p)$ living in a spacetime of $d=p+\tilde{p}+4$ dimensions.

Our general ansatz for the metric only contains an independent function $\tilde{U}(\rho)$.

$$
\begin{aligned}
ds^2_{(d)} &= e^{\frac{2}{p+1}\tilde{U}}\left[e^{\frac{2p}{p+1}r_0\rho}dt^2 - e^{-\frac{2}{p+1}r_0\rho}d\vec{y}^{\,2}_{(p)}\right] - e^{-\frac{2}{\tilde{p}+1}\tilde{U}}\gamma_{(\tilde{p}+3)\,mn}dx^m dx^n\,, \\
\gamma_{(\tilde{p}+3)\,mn}dx^m dx^n &\equiv \left[\frac{r_0}{\sinh(r_0\rho)}\right]^{\frac{2}{\tilde{p}+1}}\left[\left(\frac{r_0}{\sinh(r_0\rho)}\right)^2\frac{d\rho^2}{(\tilde{p}+1)^2} + d\Omega^2_{(\tilde{p}+2)}\right],
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- The inner horizon at $\varrho \to +\infty$ and the singularity at $\varrho = \varrho_{\rm sing} > 0$.

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$$\tilde{S} \equiv \frac{A_{\mathrm{h}\,\tilde{p}+2}}{\omega_{(\tilde{p}+2)}}$$

and T is the *Hawking temperature*

$$(2r_0)^{\frac{1}{p+1}} = \frac{4\pi}{\tilde{p}+1} T \tilde{S}^{\frac{(d-2)}{(p+1)(\tilde{p}+2)}}\,.$$

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This relation is true with the same r_0 for both inner and outer horizons.

With this formalism we will be able to compute the *entropies* of the inner (−) and outer (+) horizons and check that the product

$$\tilde{S}_+\tilde{S}_-$$

is a moduli-independent combination of conserved quantities.

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What is r_0 in more general cases?

The effective action for $\tilde{U}(\rho), \phi^i(\rho)$ is

$$I_{\rm eff}[\tilde{U},\phi^i] = \int d\tau \left\{(\dot{\tilde{U}})^2 + \tfrac{(p+1)(\tilde{p}+2)}{d-2}\mathcal{G}_{ij}\dot{\phi}^i\dot{\phi}^j - e^{2\tilde{U}}V_{\rm BB} + {r_0}^2\right\} ,$$

where we have defined the black-brane potential

$$-V_{\rm BB}(\phi,\mathcal{Q}) \equiv -\tfrac{1}{2}\mathcal{Q}^M\mathcal{Q}^N\mathcal{M}_{MN}(\phi)\, ,$$

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$$(\mathcal{Q}^M) = \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} \qquad (\mathcal{M}_{MN}) \equiv \begin{pmatrix} (I - \xi^2 R I^{-1} R)_{\Lambda\Sigma} & \xi^2 (R I^{-1})_\Lambda{}^\Sigma \\ -(I^{-1} R)^\Lambda{}_\Sigma & (I^{-1})^{\Lambda\Sigma} \end{pmatrix} ,$$

are $\mathrm{O}(n,n)$ (resp. $\mathrm{Sp}(n,n)$) vector and matrix when $\xi^2 = +1$ (resp. -1). (In general $R_{\Lambda\Sigma} = p^\Lambda = 0$ and the duality group is just $\mathrm{SO}(n)$).

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Finding a p-black brane in d dimensions with charges p, q is equivalent to solving the above mechanical system for $\tilde{U}(\rho), \phi^i(\rho)$.

We can now use the equations of motion to derive general results for black branes, generalizing those obtained by FGK for 4-dimensional black holes.

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$$\phi^i_{\rm h} = \phi^i_{\rm h}(\phi_\infty, p, q)\,, \qquad\qquad \phi^i_\infty \equiv \lim_{\rho\to 0^+} \phi^i(\rho)\,,$$

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☞ The near-horizon geometry is always $AdS_{p+2} \times S^{\tilde{p}+2}$ with the AdS_{p+2} and $S^{\tilde{p}+2}$ radii both equal to $\tilde{S}^{1/2}$.

Each critical point yields a possible extremal black-brane solution and an $AdS_{p+2} \times S^{\tilde{p}+2}$ geometry. One can go a long way in the study of extremal black holes with the attractor only, ignoring the full explicit solution.

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For $r_0 \neq 0$ one can prove the following extremality bound:

$$r_0{}^2 = \frac{[(p+1)(\tilde{p}+2)T_p + p(\tilde{p}+1)r_0]^2}{(d-2)^2} + \frac{(p+1)(\tilde{p}+2)}{(d-2)} \mathcal{G}_{ij}(\phi_\infty)\Sigma^i\Sigma^j + V_{\rm bh}(\phi_\infty, q, p)\,,$$

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However, this expression is **useless**!

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$$r_0{}^2 = \frac{[(p+1)(\tilde{p}+2)T_p + p(\tilde{p}+1)r_0]^2}{(d-2)^2} + \frac{(p+1)(\tilde{p}+2)}{(d-2)}\mathcal{G}_{ij}(\phi_\infty)\Sigma^i\Sigma^j + V_{\rm bh}(\phi_\infty, q, p)\,,$$

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In the non-extremal case we need the complete explicit solution.

3 – Construction of explicit solutions: extremal supersymmetric

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We are going to review the black holes of (ungauged) $\mathcal{N} = 2$ $d = 4$ Supergravity coupled to vector multiplets.

In order to find static extremal black holes one could try to integrate directly the equations of motion of the FGK formalism for the black-hole potential of $\mathcal{N}=2$ $d=4$ theories:

$$-V_{\rm bh} = |\mathcal{Z}|^2 + \mathcal{G}^{ij^*}\mathcal{D}_i\mathcal{Z}\mathcal{D}_{j^*}\mathcal{Z}^*\,,$$

where $\mathcal{Z}$ is the central charge of the theory

$$\mathcal{Z}(\phi,p,q) \equiv \langle\, \mathcal{V}(\phi) \mid \mathcal{Q}\,\rangle \equiv \langle \begin{pmatrix} \mathcal{L}^\Lambda \\ \mathcal{M}_\Lambda \end{pmatrix} \mid \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} \rangle \equiv p^\Lambda \mathcal{M}_\Lambda(\phi) - q_\Lambda \mathcal{L}^\Lambda(\phi)\,.$$

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There is a recipe to construct all the BPS ones.

(Behrndt, Lüst & Sabra (1997), Denef (2000), Lopes Cardoso, de Wit, Kappeli & Mohaupt (2000), Meessen, O. (2006))

1. For some complex X, define the Kähler-neutral, real, symplectic vectors $\mathcal{R}$ and $\mathcal{I}$

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2. The components of $\mathcal{I}$ are given by a symplectic vector real functions harmonic in the 3-dimensional transverse space. For single black holes ($\tau \equiv -\rho$):

$$\begin{pmatrix} \mathcal{I}^{\Lambda} \\ \\ \mathcal{I}_{\Lambda} \end{pmatrix} = \begin{pmatrix} H^{\Lambda}(\tau) \\ \\ H_{\Lambda}(\tau) \end{pmatrix} = \begin{pmatrix} H^{\Lambda}{}_{\infty} - \frac{1}{\sqrt{2}} p^{\Lambda}\tau \\ \\ H_{\Lambda\,\infty} - \frac{1}{\sqrt{2}} q_{\Lambda}\tau \end{pmatrix},$$

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5. The function $U(\tau)$ of the FGK formalism is given by

$$e^{-2U} = \langle\, \mathcal{R} \mid \mathcal{I}\,\rangle = \mathcal{I}^{\Lambda}\mathcal{R}_{\Lambda} - \mathcal{I}_{\Lambda}\mathcal{R}^{\Lambda}\,.$$

The asymptotic values of the harmonic functions, H_{∞}^{M} satisfying the condition $N = \langle\, H_{\infty} \mid \mathcal{Q}\,\rangle = 0$ have the general form

$$H^{M}{}_{\infty} = \pm\sqrt{2}\,\Im\mathrm{m}\left(\mathcal{V}_{\infty}^{M}\frac{\mathcal{Z}_{\infty}^{*}}{|\mathcal{Z}_{\infty}|}\right), \qquad \mathcal{Z}_{\infty} \equiv \mathcal{Z}(\phi_{\infty}, p, q)\,, \qquad \mathcal{V}_{\infty}^{M} \equiv \mathcal{V}^{M}(\phi_{\infty})\,.$$

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One can check in the explicit solutions all the properties predicted by the FGK formalism.

In this case the complete explicit solutions do not give much more information than the attractors, but they are going to be used as starting point for the construction of non-extremal solutions.

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$$U(\tau) = U_{\rm e}[H(\tau)]\,, \qquad Z^i(\tau) = Z^i_{\rm e}[H(\tau)]\,,$$

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Then, the non-extremal solution is given by

$$U(\tau) = U_{\rm e}[H(\tau)] + r_0\tau\,, \qquad Z^i(\tau) = Z^i{}_{\rm e}[H(\tau)]\,,$$

where now the functions H are assumed to be of the form

$$H^M = a^M + b^M\, e^{2r_0\tau}\,,$$

and the constants a^M, b^M have to be determined by explicitly solving the e.o.m.

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It has been shown that it is possible to rewrite the FGK effective action using the $H^M(\tau)$ as variables that replace $U(\tau)$ and $\phi^i(\tau)$ (Mohaupt & Waite `arXiv:0906.3451`, Mohaupt & Vaughan `arXiv:1006.3439` & `arXiv:1112.2876`, Meessen, O., Perz & Shahbazi `arXiv:1112.3332`). This confirms our hypothesis.

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More on this, later.

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5 – A complete example: $\overline{\mathbb{CP}}^n$ model

This model has n scalars Z^i that parametrize the coset space $SU(1,n)/SU(n)$. We add for convenience $Z^0 \equiv 1$, so we have

$$(Z^\Lambda) \equiv (1, Z^i)\,, \qquad (Z_\Lambda) \equiv (1, Z_i) = (1, -Z^i)\,, \qquad (\eta_{\Lambda\Sigma}) = \mathrm{diag}(+ - \cdots -)\,.$$

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It is convenient to define the complex charge combinations $\Gamma_\Lambda \equiv q_\Lambda + \frac{i}{2}\eta_{\Lambda\Sigma} p^\Sigma$.

In this model the central charge $\mathcal{Z}$, its holomorphic Kähler -covariant derivative and the black-hole potential are

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\begin{aligned}
\mathcal{Z} &= e^{\mathcal{K}/2} Z^{\Lambda}\Gamma_{\Lambda}\,, \\
\mathcal{D}_i\mathcal{Z} &= e^{3\mathcal{K}/2} Z^*_i Z^{\Lambda}\Gamma_{\Lambda} - e^{\mathcal{K}/2}\Gamma_i\,, \\
|\tilde{\mathcal{Z}}|^2 &\equiv \mathcal{G}^{ij^*}\mathcal{D}_i\mathcal{Z}\mathcal{D}_{j^*}\mathcal{Z}^* = e^{\mathcal{K}}|Z^{\Lambda}\Gamma_{\Lambda}|^2 - \Gamma^{*\,\Lambda}\Gamma_{\Lambda}\,, \\
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In $\mathcal{N}=2$ theories, in the extremal case $|\mathcal{Z}|$ plays the rôle of superpotential W. $|\tilde{\mathcal{Z}}|$ plays here the rôle of "fake" superpotential.

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We start by calculating the critical points of the black-hole potential:

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Attractor	$e^{-\mathcal{K}_{\rm h}}$	$\lvert\mathcal{Z}_{\rm h}\rvert^2$	$\lvert\tilde{\mathcal{Z}}_{\rm h}\rvert^2$	$-V_{\rm bhh}$	M
$Z^{i\,\rm susy}_{\rm h} = \Gamma^{*\,i}/\Gamma^{*\,0}$	$\Gamma^{*\,\Lambda}\Gamma_\Lambda > 0$	$\Gamma^{*\,\Lambda}\Gamma_\Lambda$	0	$\Gamma^{*\,\Lambda}\Gamma_\Lambda$	$\lvert\mathcal{Z}_\infty\rvert$
$Z^{\Lambda\,\rm nsusy}_{\rm h}\Gamma_\Lambda = 0$	$-\Gamma^{*\,\Lambda}\Gamma_\Lambda > 0$	0	$-\Gamma^{*\,\Lambda}\Gamma_\Lambda$	$-\Gamma^{*\,\Lambda}\Gamma_\Lambda$	$\lvert\tilde{\mathcal{Z}}_\infty\rvert$

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$$\mathcal{R}_\Lambda = \tfrac{1}{2}\eta_{\Lambda\Sigma}\mathcal{I}^\Sigma \,, \qquad \mathcal{R}^\Lambda = -2\eta^{\Lambda\Sigma}\mathcal{I}_\Sigma \,.$$

Then, the solutions are completely determined by the harmonic functions $H^M(\tau) = H^M - \frac{1}{\sqrt{2}}\mathcal{Q}^M\tau$ with

$$H^M{}_\infty = \pm\sqrt{2}\,\Im\mathrm{m}\left(\mathcal{V}^M_\infty \frac{\mathcal{Z}^*_\infty}{|\mathcal{Z}_\infty|}\right) \,.$$

Next, we construct the supersymmetric (extremal) solutions, associated to the supersymmetric attractor .

First we solve the stabilization equations:

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Defining, for convenience

$$\mathcal{H}_\Lambda \equiv H_\Lambda + \tfrac{i}{2}\eta_{\Lambda\Sigma}H^\Sigma \equiv e^{\mathcal{K}_\infty/2}\frac{\mathcal{Z}_\infty}{|\mathcal{Z}_\infty|}Z^*_{\Lambda\,\infty} - \frac{1}{\sqrt{2}}\Gamma_\Lambda\tau$$

the metric function and the scalars are

$$e^{-2U} = 2\mathcal{H}^{*\,\Lambda}\mathcal{H}_\Lambda \,, \qquad Z^i = \frac{\mathcal{R}^i + i\mathcal{I}^i}{\mathcal{R}^0 + i\mathcal{I}^0} = \frac{\mathcal{H}^{*i}}{\mathcal{H}^{*0}} \,.$$

Non-extremal solutions

Non-extremal solutions

Our Ansatz for the non-extremal solution is

$$e^{-2U} = e^{-2[U_{\rm e}(\mathcal{H})+r_0\tau]}\,, \qquad e^{-2U_{\rm e}(\mathcal{H})} = 2\mathcal{H}^{*\,\Lambda}\mathcal{H}_{\Lambda}\,, \qquad Z^i = Z^i{}_{\rm e}(\mathcal{H}) = \mathcal{H}^{*\,i}/\mathcal{H}^{*\,0}\,,$$

where $\mathcal{H}^{\Lambda} \equiv A^{\Lambda} + B^{\Lambda}e^{2r_0\tau}\,, \quad \Lambda = 0, \cdots, n.$

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where $\mathcal{H}^{\Lambda} \equiv A^{\Lambda} + B^{\Lambda}e^{2r_0\tau}\,, \quad \Lambda = 0, \cdots, n.$
The $2(n+1)$ complex constants A_{Λ}, B_{Λ} are found by imposing the e.o.m. ($f \equiv e^{r_0\tau}$)

$$\begin{aligned}
\ddot{U}_{\rm e} - (\dot{U}_{\rm e})^2 - \mathcal{G}_{ij^*}\dot{Z}^i\dot{Z}^{*\,j^*} &= 0\,,\\
(2r_0)^2\left[f\ddot{U}_{\rm e} + \dot{U}_{\rm e}\right] + e^{2U_{\rm e}}V_{\rm bh} &= 0\,,\\
(2r_0)^2\left[f\left(\ddot{Z}^i + \mathcal{G}^{ij^*}\partial_k\mathcal{G}_{lj^*}\dot{Z}^k\dot{Z}^l\right) + \dot{Z}^i\right] + e^{2U_{\rm e}}\mathcal{G}^{ij^*}\partial_{j^*}V_{\rm bh} &= 0\,.
\end{aligned}$$

The e.o.m. are solved if the the constants satisfy the **algebraic** equations

$$
\begin{aligned}
\Im\mathrm{m}(B^{*\,\Lambda}A_{\Lambda}) &= 0\,,\\
A^{*\,\Lambda}A^{\Sigma}\xi_{\Lambda\Sigma} &= 0\,,\\
(A^{*\,\Lambda}B^{\Sigma}+B^{*\,\Lambda}A^{\Sigma})\xi_{\Lambda\Sigma} &= 0\,,\\
B^{*\,\Lambda}B^{\Sigma}\xi_{\Lambda\Sigma} &= 0\,,\\
(2r_0)^2(B^*_i A^*_0 - B^*_0 A^*_i)A^{*\,\Lambda}A_{\Lambda} + (\Gamma^*_i A^*_0 - \Gamma^*_0 A^*_i)A^{*\,\Lambda}\Gamma_{\Lambda} &= 0\,,\\
-(2r_0)^2(B^*_i A^*_0 - B^*_0 A^*_i)B^{*\,\Lambda}B_{\Lambda} + (\Gamma^*_i B^*_0 - \Gamma^*_0 B^*_i)B^{*\,\Lambda}\Gamma_{\Lambda} &= 0\,,\\
(\Gamma^*_i A^*_0 - \Gamma^*_0 A^*_i)A^{*\,\Lambda}\Gamma_{\Lambda} + (\Gamma^*_i B^*_0 - \Gamma^*_0 B^*_i)B^{*\,\Lambda}\Gamma_{\Lambda} &= 0\,,
\end{aligned}
$$

where $\xi_{\Lambda\Sigma} \equiv 2\left(\Gamma_{\Lambda}\Gamma^*_{\Sigma} + 8{r_0}^2 A_{\Lambda}B^*_{\Sigma}\right) - \eta_{\Lambda\Sigma}\left(\Gamma^{\Omega}\Gamma^*_{\Omega} + 8{r_0}^2 A^{\Omega}B^*_{\Omega}\right)\,.$

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No differential equations remain to be solved!

Furthermore, we need to normalize the metric at spatial infinity and relate A_Λ, B_Λ to the physical parameters:

$$
\begin{aligned}
2(A^{*\,\Lambda} + B^{*\,\Lambda})(A_\Lambda + B_\Lambda) &= 1\,, \\
4\Re\mathrm{e}[B^{*\,\Lambda}(A_\Lambda + B_\Lambda)] &= 1 - M/r_0\,, \\
\frac{A^{*\,i} + B^{*\,i}}{A^{*\,0} + B^{*\,0}} &= Z^i{}_\infty\,.
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The solution can be found and it is

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\begin{aligned}
A_\Lambda &= \pm\frac{e^{\mathcal{K}_\infty/2}}{2\sqrt{2}}\left\{Z^*_{\Lambda\,\infty}\left[1 + \frac{(M^2 - e^{\mathcal{K}_\infty}|Z^{*\,\Sigma}_\infty\Gamma^*_\Sigma|^2)}{Mr_0}\right] + \frac{\Gamma_\Lambda Z^{*\,\Sigma}{}_\infty\Gamma^*_\Sigma}{Mr_0}\right\}, \\
B_\Lambda &= \pm\frac{e^{\mathcal{K}_\infty/2}}{2\sqrt{2}}\left\{Z^*_{\Lambda\,\infty}\left[1 - \frac{(M^2 - e^{\mathcal{K}_\infty}|Z^{*\,\Sigma}_\infty\Gamma^*_\Sigma|^2)}{Mr_0}\right] - \frac{\Gamma_\Lambda Z^{*\,\Sigma}_\infty\Gamma^*_\Sigma}{Mr_0}\right\},
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\end{aligned}
$$

Here $M^2{r_0}^2 = (M^2 - |\mathcal{Z}_\infty|^2)(M^2 - |\tilde{\mathcal{Z}}_\infty|^2)$, and one can show that the metric is regular in all the ${r_0}^2 > 0$ cases.

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   We get *harmonic functions with different coefficients* **non-linear in the charges!**:

$$\mathcal{H}_\Lambda \stackrel{M \to |\tilde{\mathcal{Z}}_\infty|}{\longrightarrow} \pm \frac{e^{\mathcal{K}_\infty/2}}{2\sqrt{2}} \left\{ Z^*_{\Lambda\,\infty} - \frac{1}{|\tilde{\mathcal{Z}}_\infty|} \left[-Z^*_{\Lambda\,\infty} \Gamma^{*\,\Sigma}\Gamma_\Sigma + \Gamma_\Lambda Z^{*\,\Sigma}_\infty \Gamma^*_\Sigma \right] \tau \right\} .$$

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   On the event horizon $\tau \to -\infty$ the scalars $Z^i = \mathcal{H}^{*\,i}/\mathcal{H}^{*\,0}$ take the values

$$Z^{*\,i}_{\mathrm{h}} = \frac{\Gamma^i Z^{*\,\Lambda}_\infty \Gamma^*_\Lambda - Z^{*\,i}_\infty \Gamma^{*\,\Sigma}\Gamma_\Sigma}{\Gamma^0 Z^{*\,\Gamma}_\infty \Gamma^*_\Gamma - \Gamma^{*\,\Omega}\Gamma_\Omega},$$

   which *depend manifestly on the asymptotic values*.

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which *depend manifestly on the asymptotic values*.

There is no attractor behavior in a proper sense.

The structure of the extremal non-supersymmetric solution as function of the H^Ms is the same as in the supersymmetric case.

However, no simple *substitution recipe* could have led to it.

Physical properties of the non-extremal solutions

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One can compute the "entropies" of the inner and outer horizons (event horizon (+) and Cauchy horizon (-)) at $\tau \to -\infty$ and $\tau \to +\infty$ resp.:

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$$S_\pm/\pi = \sqrt{N_{\rm R}} \pm \sqrt{N_{\rm L}}\,, \quad \Rightarrow \quad S_+S_-/\pi^2 = N_{\rm R} - N_{\rm L} \in \mathbb{Z}\,.$$

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But, even though it is suggestive, *it is not unique*. We can also write

$$S_{\pm}/\pi = \left(\sqrt{N_{\rm R}} \pm \sqrt{N_{\rm L}}\right)^2 \,,$$

with

$$N_{\rm R} \equiv M^2 - |\mathcal{Z}_\infty|^2 \,, \qquad N_{\rm L} \equiv M^2 - |\tilde{\mathcal{Z}}_\infty|^2 \,,$$

so

$$S_+S_-/\pi^2 = (N_{\rm R} - N_{\rm L})^2 \,.$$

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- If $\Gamma^{*\,\Lambda}\Gamma_\Lambda < 0$, then $|\tilde{\mathcal{Z}}_\infty| > |\mathcal{Z}_\infty|$ and the evaporation process will stop when $M = |\tilde{\mathcal{Z}}_\infty|$.

There is an attractor behavior in the evaporation process.

6 – H-FGK formalism for $\mathcal{N} = 2$, $d = 4$ supergravity

Or: Where the H^Ms come from

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$$U(\tau), Z^i(\tau)\ \ (2n_V+1) \longrightarrow \begin{pmatrix} H^\Lambda \\ H_\Lambda \end{pmatrix} \equiv H^M\,,\ \ (2n_V+2)$$

plus one constraint that should appear automatically (more on this, later).

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We introduce an auxiliary function X and proceed as in the BPS case defining the Kähler-neutral, real, symplectic vectors $\mathcal{R}^M$ and $\mathcal{I}^M$

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We know that the $\mathcal{R}^M$ can be expressed as a function of the $\mathcal{I}^M$s and vice-versa solving the *stabilization equations*. Then, we introduce *two dual sets of variables*

$$\tilde{H}_M \equiv \mathcal{R}_M \,, \qquad H^M \equiv \mathcal{I}^M \,.$$

We define the Hessian potential $\mathsf{W}(H) \equiv \tilde{H}_M(H)H^M$, or $\mathsf{W}(H) \equiv \tilde{H}_M H^M(H)$.

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Then, the FGK effective action can be written in the form

$$\begin{aligned} -I_{\rm eff}[H] &= \int d\tau \left\{\tfrac{1}{2}g_{MN}\dot{H}^M\dot{H}^N - V(H)\right\}\,, \\ g_{MN}(H) &\equiv \partial_M\partial_N \log \mathsf{W} - 2\mathsf{W}^{-2}H_M H_N\,, \\ V(H) &\equiv \left\{-\tfrac{1}{4}\partial_M\partial_N \log \mathsf{W} + \mathsf{W}^{-2}H_M H_N\right\} \mathcal{Q}^M\mathcal{Q}^N\,. \end{aligned}$$

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Then, the FGK effective action can be written in the form

$$\begin{aligned} -I_{\text{eff}}[H] &= \int d\tau \left\{ \tfrac{1}{2} g_{MN} \dot{H}^M \dot{H}^N - V(H) \right\}\,, \\ g_{MN}(H) &\equiv \partial_M \partial_N \log \mathsf{W} - 2\mathsf{W}^{-2} H_M H_N\,, \\ V(H) &\equiv \left\{ -\tfrac{1}{4} \partial_M \partial_N \log \mathsf{W} + \mathsf{W}^{-2} H_M H_N \right\} \mathcal{Q}^M \mathcal{Q}^N\,. \end{aligned}$$

All the information about the model is encoded in the Hessian potential $\mathsf{W}(H)$.

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Having the $H^M(\tau)$ that solve this action, the black-hole solution is given by

$$e^{-2U(\tau)} = \mathsf{W}[H(\tau)] \,, \qquad Z^i(\tau) = \frac{\tilde{H}^i(H) + iH^i}{\tilde{H}^0(H) + iH^0} \,.$$

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With this formalism we can try to find all the solutions in their different forms.

In the cases we have studied, all the black-hole were represented by $H^M(\tau)$ satisfying the no- NUT constraint

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$$\tilde{H}_M \left(\ddot{H}^M - {r_0}^2 H^M\right) + \left(\mathsf{W}^{-1}\dot{H}^M H_M\right)^2 = 0\,.$$

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However, some static, extremal non-supersymmetric solutions (Gimon, Larsen & Simon (2009), Galli, Goldstein, Katmadas, Perz (2011), Bossard & Katmadas (2012)) have non-harmonic H^Ms which do not satisfy the no-NUT constraint.

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This raises some questions: can they be rewritten in a harmonic form? Is there a unique way of writing in terms of H^Ms a given solution? What is their non-extremal generalization?

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First of all, there is a Freudenthal-dual representation of the same solution: it can be shown that the effective action of the H-FGK formalism is exactly invariant under the discrete transformation

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In general (this depends on the theory since $\tilde{H}_M \equiv \frac{1}{2}\partial_M \mathsf{W}$), this is a very non-linear transformation, but it is always an involution

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which shows that the physical fields

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The definition of Freudenthal duality can be extended to any symplectic vector, like the charge vector $\mathcal{Q}^M$ (Borsten, Dahanayake, Duff & Rubens (2009), Ferrara, Marrani & Yeranyan (2011)). The black-hole attractors are left invariant by it.

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$$\tilde{H}^M \frac{\delta I_{\rm eff}}{\delta H^M} = 0\,.$$

This is a *gauge identity* associated by Noether's second theorem to a gauge symmetry: multiplying by an arbitrary infinitesimal arbitrary function $f(\tau)$ and integrating over τ we find

$$\delta_f I_{\rm eff} = \int d\tau \delta_f H^M \frac{\delta I_{\rm eff}}{\delta H^M} = 0\,,$$

where we have defined the local infinitesimal transformations

$$\delta_f H^M \equiv f(\tau)\tilde{H}^M\,.$$

The finite gauge transformations can be obtained by exponentiating the infinitesimal ones:

$$\delta_f H^M \equiv f(\tau)\pounds_K H^M\,, \quad \Rightarrow \quad H'^M = e^{f(\tau)\pounds_K} H^M\,, \quad \text{with} \quad K^M(H) = \tilde{H}^M(H)\,,$$

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and the result is

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More importantly: they do not respect the no-NUT constraint, but

$$(\dot{H}^M H_M)' = -\dot{f}\mathsf{W} + \dot{H}^M H_M\,.$$

This result implies that we can bring all the static black-hole solutions of these theories to the gauge in which the no-NUT constraint is satisfied! In particular, we can do that to the Gimon-Larsen-Simon and Galli-Goldstein-Katmadas-Perz solutions.

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While this seems to suggest that in the new gauge the H^Ms will be harmonic ($\ddot{H}^M = 0$) it can be shown that there is no way to reproduce that kind of solutions with harmonic H^Ms. They are much more complicated functions. We do not know their form because the $f(\tau)$ that brings those solutions to the no-NUT gauge is very difficult to find.

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Things are much simpler in 5 dimensions!!

7 – H-FGK formalism for $\mathcal{N} = 2$, $d = 5$ supergravity

Or: Where the H^Ms come from (The 5-dimensional case)

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If we then define the derived objects

$$h_I \equiv C_{IJK}h^J h^K\,, \qquad h^I_x \equiv -\sqrt{3}\,\frac{\partial h^I}{\partial \phi^x} \qquad \text{and} \qquad h_{Ix} \equiv \sqrt{3}\,\frac{\partial h_I}{\partial \phi^x}\,,$$

we can see that they satisfy the following relations

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we can see that they satisfy the following relations

$$h^I h_I = 1 \qquad \text{and} \qquad h^I h_{Ix} = h_I h^I_x = 0\,.$$

The scalar metric g_{xy}, and the vector kinetic matrix, a_{IJ}, are given by

$$g_{xy} = h_{Ix}h^I_y \qquad \text{and} \qquad a_{IJ} = 3h_I h_J - 2C_{IJK}h^K = h_I h_J + h_{Ix}h^x_J\,.$$

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The bosonic action for $\mathcal{N}=2\ \ d=5$ supergravity with n vector supermultiplets is

$$\mathcal{I}_5 = \int_5 \left(R\star 1 + \tfrac{1}{2}g_{xy}\,d\phi^x\wedge\star d\phi^y - \tfrac{1}{2}a_{IJ}F^I\wedge\star F^J + \frac{1}{3\sqrt{3}}C_{IJK}F^I\wedge F^J\wedge A^K\right).$$

The FGK formalisms for black holes and black strings

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This theory admits black-hole ($p = 0, \tilde{p} = 1$) and black strings ($p = 1, \tilde{p} = 0$) solutions. The corresponding metric ansätze are particular cases of the general one.

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This theory admits black-hole ($p=0, \tilde{p}=1$) and black strings ($p=1, \tilde{p}=0$) solutions. The corresponding metric ansätze are particular cases of the general one.

The effective action is

$$I_{\rm eff}[\tilde{U},\phi^i] = \int d\tau \left\{ (\dot{\tilde{U}})^2 + \tfrac{(p+1)(\tilde{p}+2)}{3} g_{xy}\dot{\phi}^x\dot{\phi}^y - e^{2\tilde{U}}V_{\rm BB} + {r_0}^2 \right\} ,$$

where, in each case, we have to replace the black-brane potential $V_{\rm BB}$ by the the black-hole $V_{\rm bh}(\phi, q)$ and black-string potentials

$$\left\{ \begin{array}{rcl} -V_{\rm bh}(\phi,q) & \equiv & a^{IJ}q_I q_J = {\mathcal{Z}_{\rm e}}^2 + 3\,\partial_x\mathcal{Z}_{\rm e}\,\partial^x\mathcal{Z}_{\rm e} \,, \\ -V_{\rm bs}(\phi,p) & \equiv & a_{IJ}p^I p^J = {\mathcal{Z}_{\rm m}}^2 + 3\,\partial_x\mathcal{Z}_{\rm m}\,\partial^x\mathcal{Z}_{\rm m} \,, \end{array} \right.$$

where we have defined the *electric and magnetic central charges* by

$$\mathcal{Z}_{\rm e}(\phi,q) \equiv h^I q_I \,, \qquad \mathcal{Z}_{\rm m}(\phi,p) \equiv h_I p^I \,.$$

8 – H-variables for black holes

We replace the original variables $\tilde{U}, \phi^x$ by new ones $\tilde{H}^I$ and H_I defined by

$$\begin{aligned} e^{-\tilde{U}/2} h^I(\phi) &\equiv \tilde{H}^I\,, \\ e^{-\tilde{U}} h_I(\phi) &\equiv H_I\,, \end{aligned}$$

and the new (unconstrained) function W

$$\mathsf{W}(\tilde{H}) \equiv 2 C_{IJK} \tilde{H}^I \tilde{H}^J \tilde{H}^K\,.$$

The homogeneity properties imply that

$$\begin{aligned} e^{-\frac{3}{2}\tilde{U}} &= \tfrac{1}{2}\mathsf{W}(H)\,, \\ h_I &= (\mathsf{W}/2)^{-2/3} H_I\,, \\ h^I &= (\mathsf{W}/2)^{-1/3} \tilde{H}^I\,. \end{aligned}$$

Changing the action to the H_I variables, it becomes

$$-\tfrac{3}{2}\mathcal{I}[H] = \int d\rho \left[\partial^I \partial^J \log \mathsf{W} \left(\dot{H}_I \dot{H}_J + q_I q_J\right) - \tfrac{3}{2} {r_0}^2\right].$$

9 – K-variables for black strings

We introduce two new sets of variables, K^I and $\tilde{K}_I$, related to the original ones $(\tilde{U}, \phi^x)$ by

$$\begin{aligned} e^{-\tilde{U}} h^I(\phi) &\equiv K^I\,, \\ e^{-2\tilde{U}} h_I(\phi) &\equiv \tilde{K}_I\,, \end{aligned}$$

and the new (unconstrained) function V

$$\mathsf{V}(K) \equiv C_{IJK} K^I K^J K^K\,.$$

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Changing the action to the K^I variables, it becomes

$$-3\mathcal{I}[K] = \int d\rho \left[\partial_I \partial_J \log \mathsf{V} \left(\dot{K}^I \dot{K}^J + p^I p^J\right) - 3{r_0}^2\right].$$

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Multiplying these equations by $\dot{H}_K$ we get $\dot{\mathcal{H}} = 0$, the Hamiltonian constraint

$$\mathcal{H} \equiv \partial^I \partial^J \log \mathsf{W} \left(\dot{H}_I \dot{H}_J - q_I q_J \right) + \tfrac{3}{2} {r_0}^2 = 0 \,,$$

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Multiplying the equations of motion by H_K we obtain

$$\partial^I \log \mathsf{W} \, \ddot{H}_I = \tfrac{3}{2} {r_0}^2 \,,$$

which is the equation of $\tilde{U}$ expressed in the new variables.

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The equations of motion derived from the effective action are

$$\partial^K \partial^I \partial^J \log \mathsf{W} \left(H_I \ddot{H}_J - \dot{H}_I \dot{H}_J + q_I q_J \right) = 0 \,.$$

Multiplying these equations by $\dot{H}_K$ we get $\dot{\mathcal{H}} = 0$, the Hamiltonian constraint

$$\mathcal{H} \equiv \partial^I \partial^J \log \mathsf{W} \left(\dot{H}_I \dot{H}_J - q_I q_J \right) + \tfrac{3}{2} {r_0}^2 = 0 \,,$$

where the integration constant has been set to $\frac{3}{2}{r_0}^2$ by hand.

Multiplying the equations of motion by H_K we obtain

$$\partial^I \log \mathsf{W} \, \ddot{H}_I = \tfrac{3}{2} {r_0}^2 \,,$$

which is the equation of $\tilde{U}$ expressed in the new variables.

How useful are these new variables?

☞ In H-variables one immediately sees that, in the extremal case $r_0 = 0$

$$H_I = A_I \pm \rho q_I \,, \ \ \forall\, I \,,$$

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☞ A bit more difficult to see: in the extremal case $r_0 = 0$

$$H_I = A_I + \rho B_I \, ,$$

always solves all the equations of motion if

$$\partial^K V_{\rm bh}(B, q) = 0 \, .$$

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☞ The B_Is are called *fake charges*. Defining the *fake electric central charges*

$$\mathcal{Z}_{\rm e}(\phi, B) \equiv h^I B_I\,,$$

it is immediate to see that the following *first-order flow equations* are satisfied

$$\frac{de^{-\tilde{U}}}{d\rho} = \mathcal{Z}_{\rm e}(\phi, B)\,, \qquad\qquad \frac{d\phi^x}{d\rho} = -3e^{\tilde{U}}\partial^x\mathcal{Z}_{\rm e}(\phi, B)\,.$$

These first-order equations are extremely easy to obtain:

$$
\begin{aligned}
de^{-\tilde{U}} &= d(h^I h_I e^{-\tilde{U}}) \\
&= dh^I h_I e^{-\tilde{U}} + h^I d(h_I e^{-\tilde{U}}) \\
&= h^I d(h_I e^{-\tilde{U}}) \\
&= h^I dH_I \\
&= h^I B_I d\rho \\
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Observe that the interest of these first-order equations is merely formal since they are very difficult to integrate to obtain complete solutions.

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$$H_I = A_I \cosh r_0\rho + B_I \frac{\sinh r_0\rho}{r_0}\,.$$

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$$\begin{aligned} \partial^K\partial^I\partial^J \log \mathsf{W}(H)\left(B_I B_J - {r_0}^2 A_I A_J - q_I q_J\right) &= 0\,, \\ \partial^I\partial^J \log \mathsf{W}(H)\left(B_I B_J - {r_0}^2 A_I A_J - q_I q_J\right) &= 0\,. \end{aligned}$$

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☞ It is also possible to find all the non-extremal black holes with constant scalars of <u>all</u> the theories.

☞ Defining the new coordinate

$$\hat{\rho} \equiv \frac{\sinh(r_0 \rho)}{r_0 \cosh(r_0 \rho)}$$

we find the *first-order flow equations*

$$\frac{de^{-\tilde{U}}}{d\hat{\rho}} = \mathcal{Z}_{\mathrm{e}}(\phi, B)\,, \qquad \frac{d\phi^x}{d\hat{\rho}} = -3e^{\tilde{U}}\partial^x \mathcal{Z}_{\mathrm{e}}(\phi, B)\,.$$

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☞ The *first-order flow equations* imply the second-order e.o.m. if

$$V_{\rm bh}(\phi, B) - V_{\rm bh}(\phi, q) = {r_0}^2\,.$$

Conclusion: in any 4-dimensional, charged, static, black-hole solution of an ungauged supergravity there are two triplets of vector fields ${L^{\pm}}_m$, $m = 0, \pm 1$ given by

$$ {L^{\pm}}_1 = -\frac{e^{r_0 \pi t/S_{\pm}}}{r_0}\left(\frac{S_{\pm}}{\pi}\cosh(r_0\tau)\partial_t + \sinh(r_0\tau)\partial_\tau\right) $$

$$ {L^{\pm}}_0 = -\frac{S_{\pm}}{r_0\pi}\partial_t \,, $$

$$ {L^{\pm}}_{-1} = -\frac{e^{-r_0 \pi t/S_{\pm}}}{r_0}\left(\frac{S_{\pm}}{\pi}\cosh(r_0\tau)\partial_t - \sinh(r_0\tau)\partial_\tau\right)\,, $$

where $S_{\pm} = \frac{A_{\pm}}{4}$, which generate two $\mathfrak{sl}(2)$ algebras whose quadratic Casimirs

$$ \mathcal{H}^{\pm\,2} \equiv ({L^{\pm}}_0)^2 - \tfrac{1}{2}\left({L^{\pm}}_1 {L^{\pm}}_{-1} + {L^{\pm}}_{-1}{L^{\pm}}_1\right)\,, $$

approximate the massless Klein-Gordon equation in the two near-horizon regions:

$$ \mathcal{K}_4\Phi = \left\{-e^{-4U}{W_{-1}}^{-2}\partial_t^2 + {W_{-1}}^2\partial_\tau^2\right\}\Phi \xrightarrow{\tau\rightarrow\mp\infty} W_{-1}\left\{-\left(S_{\pm}/\pi\right)^2\partial_t^2 + \partial_\tau^2\right\}\Phi = \mathcal{H}^{\pm\,2}\Phi\,. $$

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The $\mathfrak{sl}(2)$ algebra can be extended to a complete Witt algebra, (a Virasoro algebra with no central charges):

$$L^{\pm}{}_m = -\frac{e^{mr_0\pi t/S_\pm}}{r_0}\left(\frac{S_\pm}{\pi}\cosh(mr_0\tau)\partial_t + \sinh(mr_0\tau)\partial_\tau\right) .$$

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These results can easily be extended to d-dimensional black holes using the general form of the black-hole metric etc.

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But the main question is: what is the meaning of this symmetry? (Is it really a symmetry? What of?) Can we use it to compute entropies?

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We have proven that part of our ansatz is completely general, constructing a formalism ("H-FGK") that simplifies the construction of extremal and non-extremal (black-hole and also black-string solutions in $d = 5$.

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  We have proven that part of our ansatz is completely general, constructing a formalism ("H-FGK") that simplifies the construction of extremal and non-extremal (black-hole and also black-string solutions in $d = 5$.
- ★ We have shown the power of this approach finding very general solutions and results such as the *first-order flow equations* for extremal and non-extremal objects.

- ★ We have shown that *all* the single, static, charged black holes of all ungauged supergravities have a hidden $\mathfrak{sl}(2)$ invariance that may be part of a full conformal invariance.

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- We have used the FGK formalism to construct new solutions that asymptote hvLf spacetimes, and we have shown that the near-singularity limits of known solutions also have this behaviour. Is there a holographic dual of these singularities?

We are closer to determining the general form of all single, static, black-hole and black-string solutions of $\mathcal{N}=2$,$d=4,5$ theories.