Some new results on extremal and non-extremal black holes

Tomás Ortín

(I.F.T. UAM/CSIC, Madrid)

Work done in collaboration with

- A. de Antonio (IFT-UAM/CSIC, Madrid)
- P. Galli (U. Valencia),
- P. Meessen (U. Oviedo),
- J. Perz (INFN, Padova)
- C.S. Shahbazi (IFT-UAM/CSIC, Madrid)

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Plan of the Talk:

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In this talk I will present a general ansatz and a general formalism to construct non-extremal black-hole and black-brane solutions. Then we can take their extremal non-supersymmetric limits.
I will review a complete explicit example.

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- ☞ We are going to show how we can deform any black-hole solution to get another solution whose near-horizon limit is Lifshitz-like spacetime with *hyperscaling violation*.

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☞ Inspired by this, we will also identify Lifshitz-like spacetimes with *hyperscaling violation* in the near-singularity limit of the black holes.

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We start by reviewing the FGK formalism

for black holes and black branes

in d dimensions.

2 – FGK formalism for black p-branes in d dimensions

Consider the generic d-dimensional spacetime action describing scalars ϕ^i and $(p+1)$-form potentials $A^{\Lambda}_{(p+1)}$ coupled to gravity:

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\begin{aligned}
I \;=\; & \int d^d x \sqrt{|g|}\,\Big\{R + \mathcal{G}_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j \\
& +4\tfrac{(-1)^p}{(p+2)!}\Big[I_{\Lambda\Sigma}(\phi)F^{\Lambda}_{(p+2)}\cdot F^{\Sigma}_{(p+2)} + \xi^2 R_{\Lambda\Sigma}(\phi)F^{\Lambda}_{(p+2)}\star F^{\Sigma}_{(p+2)}\Big]\Big\}\,,
\end{aligned}
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where the last term occurs only when $p = \tilde{p} = (d-4)/2$ and

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We want to find solutions describing single, static, charged, regular, black p-branes with flat worldvolume in the directions $\vec{y}_{(p)} = (y_1, \cdots, y_p)$ living in a spacetime of $d = p + \tilde{p} + 4$ dimensions.

Our general ansatz for the metric only contains an independent function $\tilde{U}(\rho)$.

$$ds^2_{(d)} = e^{\frac{2}{p+1}\tilde{U}}\left[e^{\frac{2p}{p+1}r_0\rho}dt^2 - e^{-\frac{2}{p+1}r_0\rho}d\vec{y}^{\,2}_{(p)}\right] - e^{-\frac{2}{\tilde{p}+1}\tilde{U}}\gamma_{(\tilde{p}+3)\,mn}dx^m dx^n\,,$$

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- The inner horizon at $\varrho \to +\infty$ and the singularity at $\varrho = \varrho_{\text{sing}} > 0$.

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$$\tilde{S} \equiv \frac{A_{\text{h}\,\tilde{p}+2}}{\omega_{(\tilde{p}+2)}}$$

and T is the *Hawking temperature*

$$(2r_0)^{\frac{1}{p+1}} = \frac{4\pi}{\tilde{p}+1} T \tilde{S}^{\frac{(d-2)}{(p+1)(\tilde{p}+2)}} \,.$$

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This relation is true with the same r_0 for both inner and outer horizons.

With this formalism we will be able to compute the *entropies* of the inner (−) and outer (+) horizons and check that the product

$$\tilde{S}_+ \tilde{S}_-$$

is a moduli-independent combination of conserved quantities.

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What is r_0 in more general cases?

The effective action for $\tilde{U}(\rho), \phi^i(\rho)$ is

$$I_{\rm eff}[\tilde{U}, \phi^i] = \int d\tau \left\{ (\dot{\tilde{U}})^2 + \tfrac{(p+1)(\tilde{p}+2)}{d-2} \mathcal{G}_{ij} \dot{\phi}^i \dot{\phi}^j - e^{2\tilde{U}} V_{\rm BB} + {r_0}^2 \right\} ,$$

where we have defined the black-brane potential

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$$(\mathcal{Q}^M) = \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} \qquad (\mathcal{M}_{MN}) \equiv \begin{pmatrix} (I - \xi^2 R I^{-1} R)_{\Lambda\Sigma} & \xi^2 (R I^{-1})_\Lambda{}^\Sigma \\ -(I^{-1} R)^\Lambda{}_\Sigma & (I^{-1})^{\Lambda\Sigma} \end{pmatrix} ,$$

are $\mathrm{O}(n,n)$ (resp. $\mathrm{Sp}(n,n)$) vector and matrix when $\xi^2 = +1$ (resp. -1). (In general $R_{\Lambda\Sigma} = p^\Lambda = 0$ and the duality group is just $\mathrm{SO}(n)$).

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> Finding a p-black brane in d dimensions with charges p, q is equivalent to solving the above mechanical system for $\tilde{U}(\rho), \phi^i(\rho)$.

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☞ The near-horizon geometry is always $AdS_{p+2} \times S^{\tilde{p}+2}$ with the AdS_{p+2} and $S^{\tilde{p}+2}$ radii both equal to $\tilde{S}^{1/2}$.

Each critical point yields a possible extremal black-brane solution and an $AdS_{p+2} \times S^{\tilde{p}+2}$ geometry. One can go a long way in the study of extremal black holes with the attractor only, ignoring the full explicit solution.

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$$ {r_0}^2 = \frac{[(p+1)(\tilde{p}+2)T_p + p(\tilde{p}+1)r_0]^2}{(d-2)^2} + \frac{(p+1)(\tilde{p}+2)}{(d-2)} \mathcal{G}_{ij}(\phi_\infty)\Sigma^i\Sigma^j + V_{\rm bh}(\phi_\infty, q, p)\,, $$

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In the non-extremal case we need the complete explicit solution.

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We are going to review the black holes of (ungauged) $N = 2 \;\; d = 4$ Supergravity coupled to vector multiplets.

In order to find static extremal black holes one could try to integrate directly the equations of motion of the FGK formalism for the black-hole potential of $N = 2$ $d = 4$ theories:

$$-V_{\mathrm{bh}} = |\mathcal{Z}|^2 + \mathcal{G}^{ij^*}\mathcal{D}_i\mathcal{Z}\mathcal{D}_{j^*}\mathcal{Z}^*\,,$$

where $\mathcal{Z}$ is the central charge of the theory

$$\mathcal{Z}(\phi,p,q) \equiv \langle\,\mathcal{V}(\phi)\mid\mathcal{Q}\,\rangle \equiv \langle \begin{pmatrix} \mathcal{L}^\Lambda \\ \mathcal{M}_\Lambda \end{pmatrix} \mid \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} \rangle \equiv p^\Lambda\mathcal{M}_\Lambda(\phi) - q_\Lambda\mathcal{L}^\Lambda(\phi)\,.$$

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There is a recipe to construct all the BPS ones.

(Behrndt, Lüst & Sabra (1997), Denef (2000), Lopes Cardoso, de Wit, Kappeli & Mohaupt (2000), Meessen, O. (2006))

1. For some complex X, define the Kähler-neutral, real, symplectic vectors $\mathcal{R}$ and $\mathcal{I}$

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2. The components of $\mathcal{I}$ are given by a symplectic vector real functions harmonic in the 3-dimensional transverse space. For single black holes ($\tau \equiv -\rho$):

$$\begin{pmatrix} \mathcal{I}^{\Lambda} \\ \\ \mathcal{I}_{\Lambda} \end{pmatrix} = \begin{pmatrix} H^{\Lambda}(\tau) \\ \\ H_{\Lambda}(\tau) \end{pmatrix} = \begin{pmatrix} H^{\Lambda}{}_{\infty} - \frac{1}{\sqrt{2}} p^{\Lambda}\tau \\ \\ H_{\Lambda\,\infty} - \frac{1}{\sqrt{2}} q_{\Lambda}\tau \end{pmatrix} ,$$

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4. The scalars Z^i are given by the quotients $Z^i = \dfrac{\mathcal{V}^i/X}{\mathcal{V}^0/X} = \dfrac{\mathcal{R}^i + i\mathcal{I}^i}{\mathcal{I}^0 + i\mathcal{I}^0}.$

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5. The function $U(\tau)$ of the FGK formalism is given by

$$e^{-2U} = \langle\, \mathcal{R} \mid \mathcal{I}\,\rangle = \mathcal{I}^{\Lambda}\mathcal{R}_{\Lambda} - \mathcal{I}_{\Lambda}\mathcal{R}^{\Lambda}\,.$$

The asymptotic values of the harmonic functions, H^M_∞ satisfying the condition $N = \langle\, H_\infty \mid \mathcal{Q}\,\rangle = 0$ have the general form

$$H^M{}_\infty = \pm\sqrt{2}\,\Im\mathrm{m}\left(\mathcal{V}^M_\infty \frac{\mathcal{Z}^*_\infty}{|\mathcal{Z}_\infty|}\right), \qquad \mathcal{Z}_\infty \equiv \mathcal{Z}(\phi_\infty, p, q)\,, \qquad \mathcal{V}^M_\infty \equiv \mathcal{V}^M(\phi_\infty)\,.$$

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In this case the complete explicit solutions do not give much more information than the attractors, but they are going to be used as starting point for the construction of non-extremal solutions.

4 – Construction of explicit solutions: non-extremal

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$$U(\tau) = U_{\rm e}[H(\tau)]\,, \qquad\qquad Z^i(\tau) = Z^i_{\rm e}[H(\tau)]\,,$$

where $U_{\rm e}$ and $Z^i_{\rm e}$ depend on harmonic functions $H^M(\tau) = H^M{}_\infty - \frac{1}{\sqrt{2}}\mathcal{Q}^M\tau$ given by the standard prescription for supersymmetric black holes ,

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Then, the non-extremal solution is given by

$$U(\tau) = U_{\rm e}[H(\tau)] + r_0\tau\,, \qquad Z^i(\tau) = Z^i{}_{\rm e}[H(\tau)]\,,$$

where now the functions H are assumed to be of the form

$$H^M = a^M + b^M\, e^{2r_0\tau}\,,$$

and the constants a^M, b^M have to be determined by explicitly solving the e.o.m.

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It has been shown that it is possible to rewrite the FGK effective action using the $H^M(\tau)$ as variables that replace $U(\tau)$ and $\phi^i(\tau)$ (Mohaupt & Waite arXiv:0906.3451, Mohaupt & Vaughan arXiv:1006.3439 & arXiv:1112.2876, Meessen, O., Perz & Shahbazi arXiv:1112.3332). This confirms our hypothesis.

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More on this, later.

We are going to give an explicit example, showing that one can recover both the extremal supersymmetric and non-supersymmetric black holes of a model from the general non-extremal solution found with this prescription.

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5 – A complete example: $\overline{\mathbb{CP}}^n$ model

This model has n scalars Z^i that parametrize the coset space $SU(1,n)/SU(n)$. We add for convenience $Z^0 \equiv 1$, so we have

$$(Z^\Lambda) \equiv (1, Z^i)\,, \qquad (Z_\Lambda) \equiv (1, Z_i) = (1, -Z^i)\,, \qquad (\eta_{\Lambda\Sigma}) = \text{diag}(+ - \cdots -)\,.$$

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It is convenient to define the complex charge combinations $\Gamma_\Lambda \equiv q_\Lambda + \frac{i}{2}\eta_{\Lambda\Sigma} p^\Sigma$.

In this model the central charge $\mathcal{Z}$, its holomorphic Kähler -covariant derivative and the black-hole potential are

$$
\begin{aligned}
\mathcal{Z} &= e^{\mathcal{K}/2} Z^{\Lambda} \Gamma_{\Lambda}\,, \\
\mathcal{D}_i \mathcal{Z} &= e^{3\mathcal{K}/2} Z^{*}_{i} Z^{\Lambda} \Gamma_{\Lambda} - e^{\mathcal{K}/2} \Gamma_{i}\,, \\
|\tilde{\mathcal{Z}}|^2 &\equiv \mathcal{G}^{ij^*} \mathcal{D}_i \mathcal{Z} \mathcal{D}_{j^*} \mathcal{Z}^* = e^{\mathcal{K}} |Z^{\Lambda} \Gamma_{\Lambda}|^2 - \Gamma^{*\,\Lambda} \Gamma_{\Lambda}\,, \\
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In $N=2$ theories, in the extremal case $|\mathcal{Z}|$ plays the rôle of superpotential W. $|\tilde{\mathcal{Z}}|$ plays here the rôle of “fake” superpotential.

The **extremal** case

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We start by calculating the critical points of the black-hole potential:

$$\mathcal{G}^{ij^*}\partial_{j^*}V_{\rm bh} = 2\,Z^{\Lambda}\Gamma_{\Lambda}\left(\Gamma^{*\,i} - \Gamma^{*\,0}Z^{i}\right) = 0 \quad \Rightarrow \begin{cases} Z^{i}{}_{\rm h} = \Gamma^{*\,i}/\Gamma^{*\,0}\,, \\ (\text{isolated, supersymmetric attractor}) \\ Z^{\Lambda}{}_{\rm h}\Gamma_{\Lambda} = 0\,, \\ (\text{hypersurface of non} - \text{supersymmetric} \\ \textit{attractors}) \end{cases}$$

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Attractor	$e^{-\mathcal{K}_{\rm h}}$	$\lvert\mathcal{Z}_{\rm h}\rvert^2$	$\lvert\tilde{\mathcal{Z}}_{\rm h}\rvert^2$	$-V_{\rm bhh}$	M
$Z^{i\,\rm susy}_{\rm h} = \Gamma^{*\,i}/\Gamma^{*\,0}$	$\Gamma^{*\,\Lambda}\Gamma_{\Lambda} > 0$	$\Gamma^{*\,\Lambda}\Gamma_{\Lambda}$	0	$\Gamma^{*\,\Lambda}\Gamma_{\Lambda}$	$\lvert\mathcal{Z}_{\infty}\rvert$
$Z^{\Lambda\,\rm nsusy}_{\rm h}\Gamma_{\Lambda} = 0$	$-\Gamma^{*\,\Lambda}\Gamma_{\Lambda} > 0$	0	$-\Gamma^{*\,\Lambda}\Gamma_{\Lambda}$	$-\Gamma^{*\,\Lambda}\Gamma_{\Lambda}$	$\lvert\tilde{\mathcal{Z}}_{\infty}\rvert$

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Then, the solutions are completely determined by the harmonic functions $H^M(\tau) = H^M - \frac{1}{\sqrt{2}}\mathcal{Q}^M\tau$ with

$$H^M{}_\infty = \pm\sqrt{2}\,\Im\mathrm{m}\left(\mathcal{V}^M_\infty\frac{\mathcal{Z}^*_\infty}{|\mathcal{Z}_\infty|}\right)\,.$$

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Defining, for convenience

$$\mathcal{H}_\Lambda \equiv H_\Lambda + \tfrac{i}{2}\eta_{\Lambda\Sigma}H^\Sigma \equiv e^{\mathcal{K}_\infty/2}\frac{\mathcal{Z}_\infty}{|\mathcal{Z}_\infty|}Z^*_{\Lambda\,\infty} - \frac{1}{\sqrt{2}}\Gamma_\Lambda\tau$$

the metric function and the scalars are

$$e^{-2U} = 2\mathcal{H}^{*\,\Lambda}\mathcal{H}_\Lambda\,, \qquad\qquad Z^i = \frac{\mathcal{R}^i + i\mathcal{I}^i}{\mathcal{R}^0 + i\mathcal{I}^0} = \frac{\mathcal{H}^{*i}}{\mathcal{H}^{*0}}\,.$$

Non-extremal solutions

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Our Ansatz for the non-extremal solution is

$$e^{-2U} = e^{-2[U_{\rm e}(\mathcal{H})+r_0\tau]}\,, \qquad e^{-2U_{\rm e}(\mathcal{H})} = 2\mathcal{H}^{*\,\Lambda}\mathcal{H}_{\Lambda}\,, \qquad Z^{i} = Z^{i}{}_{\rm e}(\mathcal{H}) = \mathcal{H}^{*\,i}/\mathcal{H}^{*\,0}\,,$$

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where $\mathcal{H}^\Lambda \equiv A^\Lambda + B^\Lambda e^{2r_0\tau}\,, \quad \Lambda = 0, \cdots, n.$

The $2(n+1)$ complex constants A_Λ, B_Λ are found by imposing the e.o.m. ($f \equiv e^{r_0\tau}$)

$$\begin{aligned}
\ddot{U}_{\rm e} - (\dot{U}_{\rm e})^2 - \mathcal{G}_{ij^*}\dot{Z}^i\dot{Z}^{*\,j^*} &= 0\,,\\
(2r_0)^2\left[f\ddot{U}_{\rm e} + \dot{U}_{\rm e}\right] + e^{2U_{\rm e}}V_{\rm bh} &= 0\,,\\
(2r_0)^2\left[f\left(\ddot{Z}^i + \mathcal{G}^{ij^*}\partial_k\mathcal{G}_{lj^*}\dot{Z}^k\dot{Z}^l\right) + \dot{Z}^i\right] + e^{2U_{\rm e}}\mathcal{G}^{ij^*}\partial_{j^*}V_{\rm bh} &= 0\,.
\end{aligned}$$

The e.o.m. are solved if the the constants satisfy the **algebraic** equations

$$\begin{aligned}
\Im\mathrm{m}(B^{*\,\Lambda}A_{\Lambda}) &= 0\,,\\
A^{*\,\Lambda}A^{\Sigma}\xi_{\Lambda\Sigma} &= 0\,,\\
(A^{*\,\Lambda}B^{\Sigma}+B^{*\,\Lambda}A^{\Sigma})\xi_{\Lambda\Sigma} &= 0\,,\\
B^{*\,\Lambda}B^{\Sigma}\xi_{\Lambda\Sigma} &= 0\,,\\
(2r_0)^2(B^*_iA^*_0-B^*_0A^*_i)A^{*\,\Lambda}A_{\Lambda}+(\Gamma^*_iA^*_0-\Gamma^*_0A^*_i)A^{*\,\Lambda}\Gamma_{\Lambda} &= 0\,,\\
-(2r_0)^2(B^*_iA^*_0-B^*_0A^*_i)B^{*\,\Lambda}B_{\Lambda}+(\Gamma^*_iB^*_0-\Gamma^*_0B^*_i)B^{*\,\Lambda}\Gamma_{\Lambda} &= 0\,,\\
(\Gamma^*_iA^*_0-\Gamma^*_0A^*_i)A^{*\,\Lambda}\Gamma_{\Lambda}+(\Gamma^*_iB^*_0-\Gamma^*_0B^*_i)B^{*\,\Lambda}\Gamma_{\Lambda} &= 0\,,
\end{aligned}$$

where $\xi_{\Lambda\Sigma} \equiv 2\left(\Gamma_{\Lambda}\Gamma^*_{\Sigma}+8{r_0}^2A_{\Lambda}B^*_{\Sigma}\right)-\eta_{\Lambda\Sigma}\left(\Gamma^{\Omega}\Gamma^*_{\Omega}+8{r_0}^2A^{\Omega}B^*_{\Omega}\right)\,.$

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No differential equations remain to be solved!

Furthermore, we need to normalize the metric at spatial infinity and relate A_Λ, B_Λ to the physical parameters:

$$
\begin{aligned}
2(A^{*\,\Lambda} + B^{*\,\Lambda})(A_\Lambda + B_\Lambda) &= 1\,, \\
4\Re\mathrm{e}[B^{*\,\Lambda}(A_\Lambda + B_\Lambda)] &= 1 - M/r_0\,, \\
\frac{A^{*\,i} + B^{*\,i}}{A^{*\,0} + B^{*\,0}} &= Z^i{}_\infty\,.
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\begin{aligned}
A_\Lambda &= \pm\frac{e^{\mathcal{K}_\infty/2}}{2\sqrt{2}}\left\{Z^*_{\Lambda\,\infty}\left[1+\frac{(M^2-e^{\mathcal{K}_\infty}|Z^{*\,\Sigma}_\infty\Gamma^*_\Sigma|^2)}{Mr_0}\right]+\frac{\Gamma_\Lambda Z^{*\,\Sigma}{}_\infty\Gamma^*_\Sigma}{Mr_0}\right\},\\
B_\Lambda &= \pm\frac{e^{\mathcal{K}_\infty/2}}{2\sqrt{2}}\left\{Z^*_{\Lambda\,\infty}\left[1-\frac{(M^2-e^{\mathcal{K}_\infty}|Z^{*\,\Sigma}_\infty\Gamma^*_\Sigma|^2)}{Mr_0}\right]-\frac{\Gamma_\Lambda Z^{*\,\Sigma}_\infty\Gamma^*_\Sigma}{Mr_0}\right\},
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Furthermore, we need to normalize the metric at spatial infinity and relate A_Λ, B_Λ to the physical parameters:

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Here $M^2{r_0}^2=(M^2-|\mathcal{Z}_\infty|^2)(M^2-|\tilde{\mathcal{Z}}_\infty|^2)$, and one can show that the metric is regular in all the ${r_0}^2>0$ cases.

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   We get *harmonic functions with different coefficients* **non-linear in the charges!**:

$$\mathcal{H}_\Lambda \stackrel{M \to |\tilde{\mathcal{Z}}_\infty|}{\longrightarrow} \pm \frac{e^{\mathcal{K}_\infty/2}}{2\sqrt{2}} \left\{ Z^*_{\Lambda\,\infty} - \frac{1}{|\tilde{\mathcal{Z}}_\infty|} \left[-Z^*_{\Lambda\,\infty} \Gamma^{*\,\Sigma}\Gamma_\Sigma + \Gamma_\Lambda Z^{*\,\Sigma}_\infty \Gamma^*_\Sigma \right] \tau \right\} .$$

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On the event horizon $\tau \to -\infty$ the scalars $Z^i = \mathcal{H}^{*\,i}/\mathcal{H}^{*\,0}$ take the values

$$Z^{*\,i}_{\mathrm{h}} = \frac{\Gamma^i Z^{*\,\Lambda}_\infty \Gamma^*_\Lambda - Z^{*\,i}_\infty \Gamma^{*\,\Sigma}\Gamma_\Sigma}{\Gamma^0 Z^{*\,\Gamma}_\infty \Gamma^*_\Gamma - \Gamma^{*\,\Omega}\Gamma_\Omega},$$

which *depend manifestly on the asymptotic values.*

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   which *depend manifestly on the asymptotic values*.

There is no attractor behavior in a proper sense.

The structure of the extremal non-supersymmetric solution as function of the H^Ms is the same as in the supersymmetric case.

However, no simple *substitution recipe* could have led to it.

Physical properties of the non-extremal solutions

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One can compute the "entropies" of the inner and outer horizons (event horizon (+) and Cauchy horizon (-)) at $\tau \to -\infty$ and $\tau \to +\infty$ resp.:

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$$S_{\pm}/\pi = \sqrt{N_{\rm R}} \pm \sqrt{N_{\rm L}}\,, \quad \Rightarrow \quad S_+S_-/\pi^2 = N_{\rm R} - N_{\rm L} \in \mathbb{Z}\,.$$

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But, even though it is suggestive, *it is not unique*. We can also write

$$S_{\pm}/\pi = \left(\sqrt{N_{\rm R}} \pm \sqrt{N_{\rm L}}\right)^2\,,$$

with

$$N_{\rm R} \equiv M^2 - |\mathcal{Z}_\infty|^2\,, \qquad N_{\rm L} \equiv M^2 - |\tilde{\mathcal{Z}}_\infty|^2\,,$$

so

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There is an attractor behavior in the evaporation process.

6 – H-FGK formalism for $N = 2$, $d = 5$ supergravity

Or: Where the H^Ms come from (The 5-dimensional case)

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The scalar manifold of these theories is the hypersurface in "h-space"

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If we then define the derived objects

$$h_I \equiv C_{IJK} h^J h^K\,, \qquad h^I_x \equiv -\sqrt{3}\,\frac{\partial h^I}{\partial \phi^x} \qquad \text{and} \qquad h_{Ix} \equiv \sqrt{3}\,\frac{\partial h_I}{\partial \phi^x}\,,$$

we can see that they satisfy the following relations

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The scalar metric g_{xy}, and the vector kinetic matrix, a_{IJ}, are given by

$$g_{xy} = h_{Ix} h^I_y \qquad \text{and} \qquad a_{IJ} = 3 h_I h_J - 2 C_{IJK} h^K = h_I h_J + h_{Ix} h^x_J\,.$$

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The bosonic action for $N = 2$ $d = 5$ supergravity with n vector supermultiplets is

$$\mathcal{I}_5 = \int_5 \left(R \star 1 + \tfrac{1}{2} g_{xy}\, d\phi^x \wedge \star d\phi^y - \tfrac{1}{2} a_{IJ} F^I \wedge \star F^J + \tfrac{1}{3\sqrt{3}} C_{IJK} F^I \wedge F^J \wedge A^K \right) .$$

The FGK formalisms for black holes and black strings

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This theory admits black-hole ($p = 0, \tilde{p} = 1$) and black strings ($p = 1, \tilde{p} = 0$) solutions. The corresponding metric ansätze are particular cases of the general one.

The FGK formalisms for black holes and black strings

This theory admits black-hole ($p = 0, \tilde{p} = 1$) and black strings ($p = 1, \tilde{p} = 0$) solutions. The corresponding metric ansätze are particular cases of the general one.

The effective action is

$$I_{\rm eff}[\tilde{U}, \phi^i] = \int d\tau \left\{ (\dot{\tilde{U}})^2 + \tfrac{(p+1)(\tilde{p}+2)}{3} g_{xy} \dot{\phi}^x \dot{\phi}^y - e^{2\tilde{U}} V_{\rm BB} + {r_0}^2 \right\} ,$$

where, in each case, we have to replace the black-brane potential $V_{\rm BB}$ by the the black-hole $V_{\rm bh}(\phi, q)$ and black-string potentials

$$\begin{cases} -V_{\rm bh}(\phi, q) \equiv a^{IJ} q_I q_J = {\mathcal{Z}_{\rm e}}^2 + 3\, \partial_x \mathcal{Z}_{\rm e}\, \partial^x \mathcal{Z}_{\rm e} \,, \\ -V_{\rm bs}(\phi, p) \equiv a_{IJ} p^I p^J = {\mathcal{Z}_{\rm m}}^2 + 3\, \partial_x \mathcal{Z}_{\rm m}\, \partial^x \mathcal{Z}_{\rm m} \,, \end{cases}$$

where we have defined the *electric and magnetic central charges* by

$$\mathcal{Z}_{\rm e}(\phi, q) \equiv h^I q_I \,, \qquad \mathcal{Z}_{\rm m}(\phi, p) \equiv h_I p^I \,.$$

7 – H-variables for black holes

We replace the original variables $\tilde{U}, \phi^x$ by new ones $\tilde{H}^I$ and H_I defined by

$$\begin{aligned} e^{-\tilde{U}/2} h^I(\phi) &\equiv \tilde{H}^I , \\ e^{-\tilde{U}} h_I(\phi) &\equiv H_I , \end{aligned}$$

and the new (unconstrained) function W

$$\mathsf{W}(\tilde{H}) \equiv 2 C_{IJK} \tilde{H}^I \tilde{H}^J \tilde{H}^K .$$

The homogeneity properties imply that

$$\begin{aligned} e^{-\frac{3}{2}\tilde{U}} &= \tfrac{1}{2}\mathsf{W}(H) , \\ h_I &= (\mathsf{W}/2)^{-2/3} H_I , \\ h^I &= (\mathsf{W}/2)^{-1/3} \tilde{H}^I . \end{aligned}$$

Changing the action to the H_I variables, it becomes

$$-\tfrac{3}{2}\mathcal{I}[H] = \int d\rho \left[\partial^I \partial^J \log \mathsf{W} \left(\dot{H}_I \dot{H}_J + q_I q_J \right) - \tfrac{3}{2} {r_0}^2 \right] .$$

8 – K-variables for black strings

We introduce two new sets of variables, K^I and $\tilde{K}_I$, related to the original ones $(\tilde{U}, \phi^x)$ by

$$\begin{aligned} e^{-\tilde{U}} h^I(\phi) &\equiv K^I \,, \\ e^{-2\tilde{U}} h_I(\phi) &\equiv \tilde{K}_I \,, \end{aligned}$$

and the new (unconstrained) function V

$$\mathsf{V}(K) \equiv C_{IJK} K^I K^J K^K \,.$$

The homogeneity properties imply that

$$\begin{aligned} e^{-3\tilde{U}} &= \mathsf{V}(K) \,, \\ h_I &= \mathsf{V}^{-2/3} \tilde{K}_I \,, \\ h^I &= \mathsf{V}^{-1/3} K^I \,. \end{aligned}$$

Changing the action to the K^I variables, it becomes

$$-3\mathcal{I}[K] = \int d\rho \left[\partial_I \partial_J \log \mathsf{V} \left(\dot{K}^I \dot{K}^J + p^I p^J\right) - 3{r_0}^2\right].$$

The effective actions are formally (*only formally!*) very similar. let's take the action for black holes to show how to use it.

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The equations of motion derived from the effective action are

$$\partial^K \partial^I \partial^J \log \mathsf{W} \left(H_I \ddot{H}_J - \dot{H}_I \dot{H}_J + q_I q_J \right) = 0\,.$$

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Multiplying these equations by $\dot{H}_K$ we get $\dot{\mathcal{H}} = 0$, the Hamiltonian constraint

$$\mathcal{H} \equiv \partial^I \partial^J \log \mathsf{W} \left(\dot{H}_I \dot{H}_J - q_I q_J \right) + \tfrac{3}{2} {r_0}^2 = 0\,,$$

where the integration constant has been set to $\frac{3}{2}{r_0}^2$ by hand.

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Multiplying the equations of motion by H_K we obtain

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which is the equation of $\tilde{U}$ expressed in the new variables.

How useful are these new variables?

☞ In H-variables one immediately sees that, in the extremal case $r_0 = 0$

$$H_I = A_I \pm \rho q_I \,, \;\; \forall\, I \,,$$

always solves the equations of motion in all theories.

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☞ A bit more difficult to see: in the extremal case $r_0 = 0$

$$H_I = A_I + \rho B_I \,,$$

always solves all the equations of motion if

$$\partial^K V_{\rm bh}(B, q) = 0 \,.$$

(The scalars are always $\varphi_I = H_I/H_0$ and on the horizon $\varphi_I = B_I/B_0$).

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☞ The B_Is are called *fake charges*. Defining the *fake electric central charges*

$$\mathcal{Z}_{\rm e}(\phi, B) \equiv h^I B_I \,,$$

it is immediate to see that the following *first-order flow equations* are satisfied

$$\frac{de^{-\tilde{U}}}{d\rho} = \mathcal{Z}_{\rm e}(\phi, B) \,, \qquad\qquad \frac{d\phi^x}{d\rho} = -3 e^{\tilde{U}} \partial^x \mathcal{Z}_{\rm e}(\phi, B) \,.$$

These first-order equations are extremely easy to obtain:

$$
\begin{aligned}
de^{-\tilde{U}} &= d(h^I h_I e^{-\tilde{U}}) \\
&= dh^I h_I e^{-\tilde{U}} + h^I d(h_I e^{-\tilde{U}}) \\
&= h^I d(h_I e^{-\tilde{U}}) \\
&= h^I dH_I \\
&= h^I B_I d\rho \\
&= \mathcal{Z}_{\rm e}(\phi, B) d\rho\,.
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Observe that the interest of these first-order equations is merely formal since they are very difficult to integrate to obtain complete solutions.

☞ The non-extremal case is more complicated, but we can use our *hyperbolic* ansatz

$$H_I = A_I \cosh r_0\rho + B_I \frac{\sinh r_0\rho}{r_0}\,.$$

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☞ The A_Is are easy to find, but, to find the B_Is, one has to solve the e.o.m.

$$\begin{aligned} \partial^K \partial^I \partial^J \log \mathsf{W}(H) \left(B_I B_J - {r_0}^2 A_I A_J - q_I q_J\right) &= 0\,, \\ \partial^I \partial^J \log \mathsf{W}(H) \left(B_I B_J - {r_0}^2 A_I A_J - q_I q_J\right) &= 0\,. \end{aligned}$$

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☞ It is also possible to find all the non-extremal black holes with constant scalars of <u>all</u> the theories.

☞ Defining the new coordinate

$$\hat{\rho} \equiv \frac{\sinh(r_0 \rho)}{r_0 \cosh(r_0 \rho)}$$

we find the *first-order flow equations*

$$\frac{de^{-\tilde{U}}}{d\hat{\rho}} = \mathcal{Z}_{\mathrm{e}}(\phi, B)\,, \qquad \frac{d\phi^x}{d\hat{\rho}} = -3e^{\tilde{U}}\partial^x \mathcal{Z}_{\mathrm{e}}(\phi, B)\,.$$

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☞ The *first-order flow equations* imply the second-order e.o.m. if

$$V_{\rm bh}(\phi, B) - V_{\rm bh}(\phi, q) = {r_0}^2\,.$$

9 – Hidden conformal symmetry of non-extremal black holes

In Bertini, Cacciatori and Klemm `arXiv:1106.0999` it was found that the time-radial part of the Klein-Gordon equation in the $d = 4$ background of a Schwarzschild black hole approaches the Casimir of the $\mathfrak{sl}(2)$ algebra.

This result suggests the presence of a hidden full conformal symmetry, as in the extremal Kerr case Guica, Hartman, Song, Strominger `arXiv:0809.4266`.

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Using our knowledge of the metric of a generic $d = 4$ black hole

$$
\begin{aligned}
ds^2_{(4)} &= e^{2U} dt^2 - e^{-2U} \gamma_{(-1)\,mn} dx^m dx^n \,, \\
\gamma_{(-1)\,mn} dx^m dx^n \,, &\equiv \frac{d\tau^2}{W^4_{-1}} + \frac{d\Omega^2_{-1}}{W^2_{-1}} \,, \\
d\Omega^2_{-1} &\equiv d\theta^2 + \sin^2\theta\, d\phi^2 \,, \\
W_{-1} &= \frac{\sinh r_0 \tau}{r_0} \,,
\end{aligned}
$$

we can extend this result to all the static, spherically symmetric, black holes of any ungauged supergravity (O., Shahbazi, `arXiv:1204.5910`).

In the above background, the massless Klein-Gordon equation $\Box\Phi = 0$ can be written in the form

$$e^{-2U}\partial_t^2\Phi - e^{2U}{W_{-1}}^4\partial_\tau^2\Phi - e^{2U}{W_{-1}}^2\Delta_{S^2}\Phi = 0\,,$$

where Δ_{S^2} is the Laplacian on the round S^2 of unit radius.

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Using the separation ansatz

$$\Phi = e^{-i\omega t}R(\tau)Y_m^l(\theta,\phi)\,, \quad \text{and} \quad \Delta_{S^2}Y_m^l(\theta,\phi) = -l(l+1)Y_m^l(\theta,\phi)\,,$$

we find

$$\omega^2 e^{-4U}{W_{-1}}^{-2}R(\tau) + {W_{-1}}^2\partial_\tau^2 R(\tau) = l(l+1)R(\tau)\,,$$

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Then, we can rewrite the Klein-Gordon equation as

$$\mathcal{K}_4\Phi = l(l+1)\Phi\,, \quad \text{with} \quad \mathcal{K}_4 \equiv -e^{-4U}{W_{-1}}^{-2}\partial_t^2 + {W_{-1}}^2\partial_\tau^2\,.$$

To make manifest the hidden conformal symmetry we have to Find a representation of $\mathfrak{sl}(2)$ in terms of differential operators in the $t - \tau$ submanifolds, *i.e.* find three real L_m, $m = 0, \pm 1$

$$L_m = a_{mt}(t, \tau)\partial_t + a_{m\tau}(t, \tau)\partial_\tau \, ,$$

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$$[L_m, L_n] = (m - n)L_{m+n} \, ,$$

☞ Their quadratic Casimir coincides with the differential operator $\mathcal{K}_4$

$$\mathcal{H}^2 \equiv L_0^2 - \frac{1}{2}\left(L_1 L_{-1} + L_{-1} L_1\right) = \mathcal{K}_4 \, .$$

Substituting in the equations, the ansatz

$$L_1 = l(t)\left[-m(\tau)\partial_t + n(\tau)\partial_\tau\right] \,,$$

$$L_0 = -\frac{c}{r_0}\partial_t \,,$$

$$L_{-1} = -l^{-1}(t)\left[m(\tau)\partial_t + n(\tau)\partial_\tau\right] \,.$$

we find

$$l(t) = ae^{r_0 t/c}\,, \qquad n^2(\tau) = W_{-1}{}^2\,, \qquad m(\tau) = \frac{c}{r_0}\cosh r_0\tau\,, \;\; \text{and} \;\; c^2 = \left(e^{-2U}W_{-1}{}^{-2}\right)^2 \,.$$

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The last equation is only acceptable in the two ranges of values of τ in which $e^U \sim 1/W_{-1}$: the two near-horizon regions $\tau \to \mp\infty$ in which

$$\left(e^{-2U}W_{-1}{}^{-2}\right)^2 \overset{\tau\to\mp\infty}{\sim} \left(\frac{A_\pm}{4\pi}\right)^2 + \mathcal{O}(e^{\pm r_0\tau}) = c^2 + \mathcal{O}(e^{\pm r_0\tau}) .$$

Conclusion: in any 4-dimensional, charged, static, black-hole solution of an ungauged supergravity there are two triplets of vector fields ${L^{\pm}}_m$, $m = 0, \pm 1$ given by

$$ {L^{\pm}}_1 = -\frac{e^{r_0 \pi t / S_{\pm}}}{r_0} \left(\frac{S_{\pm}}{\pi} \cosh(r_0 \tau) \partial_t + \sinh(r_0 \tau) \partial_\tau \right) $$

$$ {L^{\pm}}_0 = -\frac{S_{\pm}}{r_0 \pi} \partial_t \,, $$

$$ {L^{\pm}}_{-1} = -\frac{e^{-r_0 \pi t / S_{\pm}}}{r_0} \left(\frac{S_{\pm}}{\pi} \cosh(r_0 \tau) \partial_t - \sinh(r_0 \tau) \partial_\tau \right) \,, $$

where $S_{\pm} = \frac{A_{\pm}}{4}$, which generate two $\mathfrak{sl}(2)$ algebras whose quadratic Casimirs

$$ \mathcal{H}^{\pm\,2} \equiv ({L^{\pm}}_0)^2 - \tfrac{1}{2} \left({L^{\pm}}_1 {L^{\pm}}_{-1} + {L^{\pm}}_{-1} {L^{\pm}}_1 \right) \,, $$

approximate the massless Klein-Gordon equation in the two near-horizon regions:

$$ \mathcal{K}_4 \Phi = \left\{ -e^{-4U} {W_{-1}}^{-2} \partial_t^2 + {W_{-1}}^2 \partial_\tau^2 \right\} \Phi \xrightarrow{\tau \to \mp\infty} W_{-1} \left\{ -\left(S_{\pm}/\pi \right)^2 \partial_t^2 + \partial_\tau^2 \right\} \Phi = \mathcal{H}^{\pm\,2} \Phi \,. $$

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The $\mathfrak{sl}(2)$ algebra can be extended to a complete Witt algebra, (a Virasoro algebra with no central charges):

$$L^{\pm}{}_{m} = -\frac{e^{m r_0 \pi t / S_{\pm}}}{r_0} \left(\frac{S_{\pm}}{\pi} \cosh\left(m r_0 \tau\right) \partial_t + \sinh\left(m r_0 \tau\right) \partial_{\tau} \right) .$$

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These results can easily be extended to d-dimensional black holes using the general form of the black-hole metric etc.

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These results can easily be extended to d-dimensional black holes using the general form of the black-hole metric etc.

But the main question is: what is the meaning of this symmetry? (Is it really a symmetry? What of?) Can we use it to compute entropies?

10 – Hyperscaling-violating Lifshitz-like solutions

(Bueno, Chemissany, Meessen, O., Shahbazi, in preparation)

These solutions have spatially homogeneous metrics of the form

$$ds^2_{d+2} = \ell^2 r^{-2(d-\theta)/d}\left[r^{-2(z-1)}dt^2 - dr^2 - dx^i dx^i\right] ,$$

which are covariant under the scale transformations

$$x_i \to \lambda x_i \ , \ t \to \lambda^z t \ , \ r \to \lambda r \ , \ ds^2_{d+2} \to \lambda^{2\theta/d} ds^2_{d+2} \, ,$$

where λ is a dimensionless parameter, ℓ is the Lifshitz radius, z is the dynamical critical exponent and θ is the hyperscaling violating exponent.

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10 – Hyperscaling-violating Lifshitz-like solutions

(Bueno, Chemissany, Meessen, O., Shahbazi, in preparation)

These solutions have spatially homogeneous metrics of the form

$$ds^2_{d+2} = \ell^2 r^{-2(d-\theta)/d} \left[r^{-2(z-1)} dt^2 - dr^2 - dx^i dx^i \right] ,$$

which are covariant under the scale transformations

$$x_i \to \lambda x_i \ , \ t \to \lambda^z t \ , \ r \to \lambda r \ , \ ds^2_{d+2} \to \lambda^{2\theta/d} ds^2_{d+2} \, ,$$

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☞ The metrics with $\theta \neq 0$ hyperscaling-violating Lifshitz-like metrics (hvLf) are holographically related to theories in which the would-be scale symmetry is violated.

We are going to construct hvLf metrics using the FGK formalism and the following observation: if we use the metrics

$$
\begin{aligned}
ds^2_{(4)} &= e^{2U}dt^2 - e^{-2U}\gamma_{\kappa\, mn}dx^m dx^n\,, \\
\gamma_{\kappa\, mn}dx^m dx^n\,, &\equiv \frac{d\tau^2}{W_\kappa{}^4} + \frac{d\Omega^2_\kappa}{W^2_\kappa}\,,
\end{aligned}
$$

with $d\Omega^2_\kappa, W_\kappa$ given by one of these three cases

$$
\begin{aligned}
d\Omega^2_{-1} &\equiv d\vartheta^2 + \sin^2\vartheta\, d\phi^2\,, & W_{-1} &= \frac{\sinh r_0\tau}{r_0}\,, \\
d\Omega^2_{+1} &\equiv d\vartheta^2 + \sinh^2\vartheta\, d\phi^2\,, & W_{1} &= \frac{\cosh r_0\tau}{r_0}\,, \\
d\Omega^2_{0} &\equiv d\vartheta^2 + d\phi^2\,, & W^{\pm}_{0} &= a e^{\mp r_0\tau}\,,
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the effective equations of motion satisfied by $U(\tau)$ and $\phi^i(\tau)$ are the same!

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the effective equations of motion satisfied by $U(\tau)$ and $\phi^i(\tau)$ are the same!

Then, using $U(\tau)$ and $\phi^i(\tau)$ from a black-hole solution ($\kappa = -1$) we can get three new solutions. We are going to consider only the $\kappa = 0$ ones.

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In the spherically-symmetric case the spacetime metric approaches a product of a 2-dimensional Rindler metric $\mathcal{R}i^2$ and a 2-sphere S^2 of area $4S_\pm$. Both horizons satisfy $r_0 = 2S_\pm T_\pm$

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☞ e^{-2U} vanishes for some value of $\tau_{\rm s} \in (0, +\infty)$ at which the physical singularity of the black-hole lies. **The generic behaviour of the black-hole metric near the singularities has not been studied**. We have to do it case by case.

We can construct two $\kappa = 0$ metrics, but we only study one:

$$ds^2_{(-)} = e^{2U} dt^2 - e^{-2U} \left[e^{-4 r_0 \tau} {r_0}^4 d\tau^2 + e^{-2 r_0 \tau} {r_0}^2 \left(d\vartheta^2 + d\phi^2 \right) \right] .$$

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In the other near-horizon limit$\tau \to +\infty$ the metric approaches $\mathcal{R}i^2 \times \mathbb{R}^2$.

To study a near-singularity limit we consider the solution whose e^{-2U} is that of the usual Reissner-Nordström black hole. In the usual coordinates, the new solution is

$$ds^2_{(\pm)} = \frac{(r-r_+)(r-r_-)}{r^2}dt^2 - \frac{{r_0}^4 r^2}{(r-r_\pm)(r-r_\mp)^5}dr^2 - \frac{{r_0}^2 r^2}{(r-r_\mp)^2}(d\vartheta^2 + d\phi^2)\,.$$

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Actually, in the $r \to 0$ limit, the behaviour of this metric is analogous to that of the standard Reissner-Nordström black hole in a small patch around $\vartheta = \pi/2$ where $d\vartheta^2 + \sin^2\vartheta d\phi^2 \sim d\vartheta^2 + d\phi^2$:

$$ds^2 = \frac{(r-r_+)(r-r_-)}{r^2}dt^2 - \frac{r^2}{(r-r_+)(r-r_-)}dr^2 - r^2(d\vartheta^2 + d\phi^2)$$

$$\sim \frac{r_+ r_-}{r^2}dt^2 - \frac{r^2}{r_+ r_-}dr^2 - r^2(d\vartheta^2 + d\phi^2)\,,$$

i.e. hvLf with $z = 3$, $\theta = 4$ and $\ell = \sqrt{r_+ r_-}$.

We can also take the near-horizon limit of the Schwarzschild metric with negative mass (and a naked, timelike singularity) in a neighborhood of $\vartheta = \pi/2$:

$$ds^2 = \left(1 + \frac{2|M|}{r}\right) dt^2 - \left(1 + \frac{2|M|}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + d\phi^2)$$

$$\sim \frac{2|M|}{r} dt^2 - \frac{r}{2|M|} dr^2 - r^2(d\vartheta^2 + d\phi^2)\,,$$

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This suggests the possibility of finding a quantum system dual to these singularities...

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  We have proven that part of our ansatz is completely general, constructing a formalism ("H-FGK") that simplifies the construction of extremal and non-extremal (black-hole and also black-string solutions in $d = 5$.
- We have shown the power of this approach finding very general solutions and results such as the *first-order flow equations* for extremal and non-extremal objects.

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- We have used the FGK formalism to construct new solutions that asymptote hvLf spacetimes, and we have shown that the near-singularity limits of known solutions also have this behaviour. Is there a holographic dual of these singularities?

We are closer to determining the general form of all single, static, black-hole and black-string solutions of $N = 2$,$d = 4, 5$ theories.