Non-extremal black holes and branes of N=2, d=4,5 Supergravity

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Work done in collaboration with

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Plan of the Talk:

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In this talk I will present a general ansatz and a general formalism to construct non-extremal black-hole and black-brane solutions and we will study some examples. First, we will review some general facts.

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Algebraic approach:
- Ferrara, Gibbons & Kallosh, (1997) (general formalism)
- Ceresole & Dall′Agata (2007) ("fake"superpotentials)

Explicit solutions:

Supersymmetric (*i.e.* extremal) :
- Tod (1983) (pure $N = 2$)
- Behrndt, Luest & Sabra (1997)($N = 2$ + Vs.)
- Caldarelli & Klemm (2003) (pure gauged $N = 2$)
- Huebscher, Meessen, O. & Vaula (2007), Meessen, (2008) ($N = 2$ + Vs non − Abelian − gauged)
- Cacciatori, Klemm, Mansi & Zorzan (2008) ($N = 2$ + Vs Abelian − gaug
- Meessen, O. & Vaula (2010) (all $N \geq 2$)

Non − extremal :
- Cvetic & Youm (1996)
- O. (1996)
- Kastor & Win (1996)
- Mohaupt & Vaughan (2010) (general Ansatz $d = 5$)
- Galli, O., Perz & Shahbazi (2011) (general Ansatz $d = 4$)

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We start by reviewing the FGK formalism for black holes in $d = 4$.

Later, we will generalize it to black branes in any dimension.

2 – FGK formalism for $d = 4$ black holes

Ferrara, Gibbons and Kallosh (1997) considered the general 4-dimensional action

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\begin{aligned}
I &= \int d^4x \sqrt{|g|}\,\{R + \mathcal{G}_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j \\
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They also considered the general metric for any static non-extremal black hole

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ds^2 = e^{2U(\tau)}dt^2 - e^{-2U(\tau)}\left[\frac{{r_0}^4}{\sinh^4 r_0\tau}d\tau^2 + \frac{{r_0}^2}{\sinh^2 r_0\tau}d\Omega^2_{(2)}\right] .
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- ☞ What is r_0 like for more general black holes?

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To determine completely the metric of any static, regular, spherically symmetric black hole we only need to find the function $U(\tau)$.

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The latter can be integrated out so they are effectively replaced by the electric, q_Λ, and magnetic, p^Λ charges. The general system reduces to an effective mechanical system with variables $U(\tau)$, $\phi^i(\tau)$:

$$I_{\text{eff}}[U,\phi^i] = \int d\tau \left\{(U')^2 + \tfrac{1}{2}\mathcal{G}_{ij}\phi^{i\,\prime}\phi^{j\,\prime} - e^{2U}V_{\text{bh}} + r_0{}^2\right\} \,,$$

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where FGK defined the black-hole potential

$$-V_{\rm bh}(\phi,q,p) \equiv -\tfrac{1}{2}\begin{pmatrix} p^{\Lambda} & q_{\Lambda}\end{pmatrix} \begin{pmatrix} (I+RI^{-1}R)_{\Lambda\Sigma} & -(RI^{-1})_{\Lambda}{}^{\Sigma} \\ -(I^{-1}R)^{\Lambda}{}_{\Sigma} & (I^{-1})^{\Lambda\Sigma} \end{pmatrix} \begin{pmatrix} p^{\Sigma} \\ q_{\Sigma} \end{pmatrix} \, ,$$

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Finding a black hole with charges p, q is equivalent to solving the above mechanical system for $U(\tau), \phi^i(\tau)$.

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Each critical point yields a possible extremal black-hole solution and an $AdS_2 \times S^2$ geometry. One can go a long way with the attractor only, ignoring the full explicit solution.

In the general case one can prove the following extremality bound:

$$r_0{}^2 = M^2 + \tfrac{1}{2}\mathcal{G}_{ij}(\phi_\infty)\Sigma^i\Sigma^j + V_{\rm bh}(\phi_\infty, q, p)\,, \geq 0\,,$$

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We need to find the complete explicit solution in the non-extremal case.

Whenever we can write $-\left[e^{2U}V_{\rm bh} - {r_0}^2\right] = (\partial_U Y)^2 + 2\,\mathcal{G}^{ij}\partial_i Y \partial_j Y$ for some *(generalized) superpotential* $Y(U, \phi^i, p, q, r_0)$, we can rewrite the effective action as

$$I_{\rm eff}[U, \phi^i] = \int d\tau \left\{(U' - \partial_U Y)^2 + \tfrac{1}{2}\mathcal{G}_{ij}(\phi^{i\,\prime} - 2\,\mathcal{G}^{ik}\partial_k Y)(\phi^{j\,\prime} - 2\,\mathcal{G}^{jl}\partial_l Y) + 2Y'\right\} .$$

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A generalized superpotential $Y(U, \phi^i, p, q, r_0)$ exists in all theories whose scalar manifold (after timelike dimensional reduction) is a symmetric coset space (in particular for all $N > 2$ supergravities) (Andrianopoli, D'Auria, Orazi & Trigiante (2009), Chemissany, Fré, Rosseel, Sorin, Trigiante & Van Riet (2010)).

In the extremal case $r_0 = 0$, if there is a generalized superpotential $Y(U, \phi^i, p, q)$, it factorizes

$$Y(U, \phi^i, p, q) = e^U W(\phi^i, p, q)\,,$$

where $W(\phi^i, p, q)$ is called the *superpotential*, and the flow equations take the form (Ceresole & Dall'Agata (2007))

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The stationary values of the superpotential $\partial_i W|_{\phi_{\rm h}} = 0$ give the the entropy:

$$S = \pi |W(\phi_{\rm h}, p, q)|^2\,,$$

while the mass is

$$M = |W(\phi_\infty, p, q)|\,.$$

3 – Direct construction of solutions: extremal supersymmetric

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We are going to review the example of (ungauged) $N = 2$ Supergravity coupled to vector multiplets.

4 − $N = 2, d = 4$ ungauged SUGRA coupled to vector multiplets

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The field content

$4 - N = 2, d = 4$ ungauged SUGRA coupled to vector multiplets

<u>The field content</u>

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Hypermultiplets can be ignored for black-hole solutions.

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Local $N = 2$ supersymmetry requires the Kähler-Hodge manifold to be a special Kähler manifold, so it is the base space of a $2(n_V + 1)$-dimensional vector bundle with $Sp[2(n_V + 1), \mathbb{R}]$ structure group, on which we can define the constrained symplectic section

$$\mathcal{V} = \begin{pmatrix} \mathcal{L}^\Lambda(Z, Z^*) \\ \mathcal{M}_\Lambda(Z, Z^*) \end{pmatrix}.$$

$\mathcal{V}$ can be thought of as just a redundant description of the physical scalars with manifest symplectic symmetry, which also acts on the electric and magnetic charges:

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The action of the bosonic fields of the ungauged theory is of the general FGK form:

$$\begin{aligned} S &= \int d^4x \sqrt{|g|}\, \left[R + 2\mathcal{G}_{ij^*}\partial_{\mu}Z^i \partial^{\mu} Z^{*\,j^*} + 2\Im\text{m}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda\,\mu\nu} F^{\Sigma}{}_{\mu\nu} \right. \\ & \left. \qquad -2\Re\text{e}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda\,\mu\nu} \star F^{\Sigma}{}_{\mu\nu}\right]\,, \Rightarrow\; -V_{\text{bh}} = |\mathcal{Z}|^2 + \mathcal{G}^{ij^*}\mathcal{D}_i\mathcal{Z}\mathcal{D}_{j^*}\mathcal{Z}^*\,. \end{aligned}$$

In order to find static extremal black holes one could try to integrate directly the equations of motion of the FGK formalism for the black-hole potential of $N = 2$ $d = 4$ theories:

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There is a recipe to construct all the BPS ones:
(Denef (2000), Behrndt, Lüst & Sabra (1997), Meessen, O. (2006))

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2. The components of $\mathcal{I}$ are given by a symplectic vector real functions harmonic in the 3-dimensional transverse space. For single black holes :

$$\begin{pmatrix} \mathcal{I}^{\Lambda} \\ \\ \mathcal{I}_{\Lambda} \end{pmatrix} = \begin{pmatrix} H^{\Lambda}(\tau) \\ \\ H_{\Lambda}(\tau) \end{pmatrix} = \begin{pmatrix} H^{\Lambda}{}_{\infty} - \frac{1}{\sqrt{2}} p^{\Lambda}\tau \\ \\ H_{\Lambda\,\infty} - \frac{1}{\sqrt{2}} q_{\Lambda}\tau \end{pmatrix},$$

with no sources of NUT charge, *i.e.* $\langle\, H_{\infty} \mid \mathcal{Q}\,\rangle = H^{\Lambda}{}_{\infty} q_{\Lambda} - H_{\Lambda\infty} p^{\Lambda} = 0$

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$$\begin{pmatrix} \mathcal{I}^{\Lambda} \\ \\ \mathcal{I}_{\Lambda} \end{pmatrix} = \begin{pmatrix} H^{\Lambda}(\tau) \\ \\ H_{\Lambda}(\tau) \end{pmatrix} = \begin{pmatrix} H^{\Lambda}{}_{\infty} - \frac{1}{\sqrt{2}} p^{\Lambda}\tau \\ \\ H_{\Lambda\,\infty} - \frac{1}{\sqrt{2}} q_{\Lambda}\tau \end{pmatrix},$$

with no sources of NUT charge, *i.e.* $\langle\, H_{\infty} \mid \mathcal{Q}\,\rangle = H^{\Lambda}{}_{\infty} q_{\Lambda} - H_{\Lambda\infty} p^{\Lambda} = 0$

3. $\mathcal{R}$ is to be found from $\mathcal{I}$ by solving the generalized *stabilization equations*.

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5. The function $U(\tau)$ of the FGK formalism is given by

$$e^{-2U} = \langle\, \mathcal{R} \mid \mathcal{I}\,\rangle = \mathcal{I}^{\Lambda}\mathcal{R}_{\Lambda} - \mathcal{I}_{\Lambda}\mathcal{R}^{\Lambda}\,.$$

The asymptotic values of the harmonic functions, H^M_∞ satisfying the condition $N = \langle H_\infty \mid \mathcal{Q} \rangle = 0$ have the general form

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In this case the complete explicit solutions do not give much more information than the algebraic approach, but they are going to be used as starting point for the construction of non-extremal solutions later on.

5 – Direct construction of solutions: non-extremal

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$$U(\tau) = U_{\rm e}[H(\tau)]\,, \qquad\qquad Z^i(\tau) = Z^i_{\rm e}[H(\tau)]\,,$$

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Then, the non-extremal solution is given by

$$U(\tau) = U_{\rm e}[H(\tau)] + r_0\tau\,, \qquad Z^i(\tau) = Z^i{}_{\rm e}[H(\tau)]\,,$$

where now the functions H are assumed to be of the form

$$H^M = a^M + b^M\, e^{2r_0\tau}\,,$$

and the constants a^M, b^M have to be determined by explicitly solving the e.o.m.

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> It has been shown that it is possible to rewrite the FGK effective action using the $H^M(\tau)$ as variables that replace $U(\tau)$ and $\phi^i(\tau)$ (Mohaupt & Waite `arXiv:0906.3451`, Mohaupt & Vaughan `arXiv:1006.3439` & `arXiv:1112.2876`, Meessen, O., Perz & Shahbazi `arXiv:1112.3332`). This confirms our hypothesis.

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More on this, later.

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6 – A complete example: $\overline{\mathbb{CP}}^n$ model

This model and has n scalars Z^i that parametrize the coset space $SU(1,n)/SU(n)$. We add for convenience $Z^0 \equiv 1$, so we have

$$(Z^\Lambda) \equiv (1, Z^i)\,, \qquad (Z_\Lambda) \equiv (1, Z_i) = (1, -Z^i)\,, \qquad (\eta_{\Lambda\Sigma}) = \mathrm{diag}(+ - \cdots -)\,.$$

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The covariantly holomorphic symplectic section reads $\mathcal{V} = e^{\mathcal{K}/2}\begin{pmatrix} Z^\Lambda \\ \\ -\frac{i}{2} Z_\Lambda \end{pmatrix}$.

It is convenient to define the complex charge combinations $\Gamma_\Lambda \equiv q_\Lambda + \frac{i}{2}\eta_{\Lambda\Sigma} p^\Sigma$.

The central charge $\mathcal{Z}$, its holomorphic Kähler -covariant derivative and the black-hole potential are given by

$$\begin{aligned} \mathcal{Z} &= e^{\mathcal{K}/2} Z^{\Lambda} \Gamma_{\Lambda} \,, \\ \mathcal{D}_i \mathcal{Z} &= e^{3\mathcal{K}/2} Z_i^{*} Z^{\Lambda} \Gamma_{\Lambda} - e^{\mathcal{K}/2} \Gamma_i \,, \\ |\tilde{\mathcal{Z}}|^2 &\equiv \mathcal{G}^{ij^*} \mathcal{D}_i \mathcal{Z} \mathcal{D}_{j^*} \mathcal{Z}^* = e^{\mathcal{K}} |Z^{\Lambda} \Gamma_{\Lambda}|^2 - \Gamma^{*\,\Lambda} \Gamma_{\Lambda} \,, \\ -V_{\rm bh} &= |\mathcal{Z}|^2 + |\tilde{\mathcal{Z}}|^2 \,. \end{aligned}$$

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Remember that in $N=2$ theories, in the extremal case $|\mathcal{Z}|$ plays the rôle of superpotential W. In this case $|\tilde{\mathcal{Z}}|$ will play the rôle of “fake” superpotential.

In this case we can write

$$-\left[e^{2U}V_{\rm bh} - {r_0}^2\right] = \Upsilon^2 + 4\,\mathcal{G}^{ij^*}\,\Psi_i\Psi^*_{j^*}\,,$$

$$\Upsilon = \frac{e^U}{\sqrt{2}}\sqrt{|\mathcal{Z}|^2 + |\tilde{\mathcal{Z}}|^2 + e^{-2U}{r_0}^2 + \sqrt{\left(|\mathcal{Z}|^2 + |\tilde{\mathcal{Z}}|^2 + e^{-2U}{r_0}^2\right)^2 - 4|\mathcal{Z}|^2|\tilde{\mathcal{Z}}|^2}}\,,$$

$$\Psi_i = e^{2U}\,\frac{\mathcal{Z}^*\,\mathcal{D}_i\mathcal{Z}}{\Upsilon}\,,$$

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$$\Psi_i = e^{2U}\frac{\mathcal{Z}^*\,\mathcal{D}_i\mathcal{Z}}{\Upsilon}\,,$$

Since

$$\partial_U\Psi_i-\partial_i\Upsilon=\partial_i\Psi_j-\partial_j\Psi_i=\partial_{i*}\Psi_j-\partial_j\Psi^*_{i^*}=0\,,$$

there exists a generalized superpotential, whose gradient generates the vector field $(\Upsilon,\Psi_i,\Psi^*_{j^*})$ and the first-order equations

$$U'=\Upsilon\,,\qquad\qquad Z^{i\,\prime}=2\,\mathcal{G}^{ij^*}\,\Psi^*_{j^*}\,.$$

but it is very difficult to find explicitly.

The extremal case

The **extremal** case

We start by calculating the critical points of the black-hole potential:

$$\mathcal{G}^{ij^*}\partial_{j^*}V_{\rm bh} = 2\,Z^{\Lambda}\Gamma_{\Lambda}\left(\Gamma^{*\,i} - \Gamma^{*\,0}Z^{i}\right) = 0 \quad \Rightarrow \left\{ \begin{array}{l} Z^{i}{}_{\rm h} = \Gamma^{*\,i}/\Gamma^{*\,0}\,, \\ \text{(isolated, supersymmetric attractor)} \\ \\ Z^{\Lambda}{}_{\rm h}\Gamma_{\Lambda} = 0\,, \\ \text{(non} - \text{supersymmetric hypersurface)} \end{array} \right.$$

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Attractor	$e^{-\mathcal{K}_{\rm h}}$	$\lvert\mathcal{Z}_{\rm h}\rvert^2$	$\lvert\tilde{\mathcal{Z}}_{\rm h}\rvert^2$	$-V_{\rm bhh}$	M
$Z^{i\,\rm susy}_{\rm h} = \Gamma^{*\,i}/\Gamma^{*\,0}$	$\Gamma^{*\,\Lambda}\Gamma_{\Lambda} > 0$	$\Gamma^{*\,\Lambda}\Gamma_{\Lambda}$	0	$\Gamma^{*\,\Lambda}\Gamma_{\Lambda}$	$\lvert\mathcal{Z}_{\infty}\rvert$
$Z^{\Lambda\,\rm nsusy}_{\rm h}\Gamma_{\Lambda} = 0$	$-\Gamma^{*\,\Lambda}\Gamma_{\Lambda} > 0$	0	$-\Gamma^{*\,\Lambda}\Gamma_{\Lambda}$	$-\Gamma^{*\,\Lambda}\Gamma_{\Lambda}$	$\lvert\tilde{\mathcal{Z}}_{\infty}\rvert$

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Then, the solutions are completely determined by the harmonic functions $H^M(\tau) = H^M - \frac{1}{\sqrt{2}}\mathcal{Q}^M\tau$ with

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$$H^M{}_\infty = \pm\sqrt{2}\,\Im\mathrm{m}\left(\mathcal{V}^M_\infty\frac{\mathcal{Z}^*_\infty}{|\mathcal{Z}_\infty|}\right)\,.$$

Defining, for convenience's sake

$$\mathcal{H}_\Lambda \equiv H_\Lambda + \tfrac{i}{2}\eta_{\Lambda\Sigma}H^\Sigma \equiv e^{\mathcal{K}_\infty/2}\frac{\mathcal{Z}_\infty}{|\mathcal{Z}_\infty|}Z^*_{\Lambda\,\infty} - \frac{1}{\sqrt{2}}\Gamma_\Lambda\tau$$

the metric function and the scalars are

$$e^{-2U} = 2\mathcal{H}^{*\,\Lambda}\mathcal{H}_\Lambda\,, \qquad\qquad Z^i = \frac{\mathcal{R}^i + i\mathcal{I}^i}{\mathcal{R}^0 + i\mathcal{I}^0} = \frac{\mathcal{H}^{*i}}{\mathcal{H}^{*0}}\,.$$

Non-extremal solutions

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Our Ansatz for the non-extremal solution is

$$e^{-2U} = e^{-2[U_{\rm e}(\mathcal{H})+r_0\tau]}\,, \qquad e^{-2U_{\rm e}(\mathcal{H})} = 2\mathcal{H}^{*\,\Lambda}\mathcal{H}_{\Lambda}\,, \qquad Z^i = Z^i{}_{\rm e}(\mathcal{H}) = \mathcal{H}^{*\,i}/\mathcal{H}^{*\,0}\,,$$

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where $\mathcal{H}^{\Lambda} \equiv A^{\Lambda} + B^{\Lambda}e^{2r_0\tau}\,, \quad \Lambda = 0,\cdots,n.$

The $2(n+1)$ complex constants A_{Λ}, B_{Λ} are found by imposing the e.o.m. $(f \equiv e^{r_0\tau})$

$$\begin{aligned} \ddot{U}_{\rm e} - (\dot{U}_{\rm e})^2 - \mathcal{G}_{ij^*}\dot{Z}^i\dot{Z}^{*\,j^*} &= 0\,, \\ (2r_0)^2\left[f\ddot{U}_{\rm e} + \dot{U}_{\rm e}\right] + e^{2U_{\rm e}}V_{\rm bh} &= 0\,, \\ (2r_0)^2\left[f\left(\ddot{Z}^i + \mathcal{G}^{ij^*}\partial_k\mathcal{G}_{lj^*}\dot{Z}^k\dot{Z}^l\right) + \dot{Z}^i\right] + e^{2U_{\rm e}}\mathcal{G}^{ij^*}\partial_{j^*}V_{\rm bh} &= 0\,. \end{aligned}$$

The e.o.m. are solved if the the constants satisfy the **algebraic** equations

$$
\begin{aligned}
\Im\mathrm{m}(B^{*\,\Lambda}A_{\Lambda}) &= 0\,,\\
A^{*\,\Lambda}A^{\Sigma}\xi_{\Lambda\Sigma} &= 0\,,\\
(A^{*\,\Lambda}B^{\Sigma}+B^{*\,\Lambda}A^{\Sigma})\xi_{\Lambda\Sigma} &= 0\,,\\
B^{*\,\Lambda}B^{\Sigma}\xi_{\Lambda\Sigma} &= 0\,,\\
(2r_0)^2(B^*_i A^*_0 - B^*_0 A^*_i)A^{*\,\Lambda}A_{\Lambda} + (\Gamma^*_i A^*_0 - \Gamma^*_0 A^*_i)A^{*\,\Lambda}\Gamma_{\Lambda} &= 0\,,\\
-(2r_0)^2(B^*_i A^*_0 - B^*_0 A^*_i)B^{*\,\Lambda}B_{\Lambda} + (\Gamma^*_i B^*_0 - \Gamma^*_0 B^*_i)B^{*\,\Lambda}\Gamma_{\Lambda} &= 0\,,\\
(\Gamma^*_i A^*_0 - \Gamma^*_0 A^*_i)A^{*\,\Lambda}\Gamma_{\Lambda} + (\Gamma^*_i B^*_0 - \Gamma^*_0 B^*_i)B^{*\,\Lambda}\Gamma_{\Lambda} &= 0\,,
\end{aligned}
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where $\xi_{\Lambda\Sigma} \equiv 2\left(\Gamma_{\Lambda}\Gamma^*_{\Sigma} + 8{r_0}^2 A_{\Lambda}B^*_{\Sigma}\right) - \eta_{\Lambda\Sigma}\left(\Gamma^{\Omega}\Gamma^*_{\Omega} + 8{r_0}^2 A^{\Omega}B^*_{\Omega}\right)\,.$

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No differential equations remain to be solved!

Furthermore, we need to normalize the metric at spatial infinity and relate A_Λ, B_Λ to the physical parameters:

$$
\begin{aligned}
2(A^{*\,\Lambda} + B^{*\,\Lambda})(A_\Lambda + B_\Lambda) &= 1\,, \\
4\Re\mathrm{e}[B^{*\,\Lambda}(A_\Lambda + B_\Lambda)] &= 1 - M/r_0\,, \\
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Up to a phase to be determined in the supersymmetric extremal limit the solution is

$$
\begin{aligned}
A_\Lambda &= \pm\frac{e^{\mathcal{K}_\infty/2}}{2\sqrt{2}}\left\{Z^*_{\Lambda\,\infty}\left[1 + \frac{(M^2 - e^{\mathcal{K}_\infty}|Z^{*\,\Sigma}_\infty \Gamma^*_\Sigma|^2)}{M r_0}\right] + \frac{\Gamma_\Lambda Z^{*\,\Sigma}{}_\infty \Gamma^*_\Sigma}{M r_0}\right\}, \\
B_\Lambda &= \pm\frac{e^{\mathcal{K}_\infty/2}}{2\sqrt{2}}\left\{Z^*_{\Lambda\,\infty}\left[1 - \frac{(M^2 - e^{\mathcal{K}_\infty}|Z^{*\,\Sigma}_\infty \Gamma^*_\Sigma|^2)}{M r_0}\right] - \frac{\Gamma_\Lambda Z^{*\,\Sigma}_\infty \Gamma^*_\Sigma}{M r_0}\right\},
\end{aligned}
$$

Furthermore, we need to normalize the metric at spatial infinity and relate A_Λ, B_Λ to the physical parameters:

$$
\begin{aligned}
2(A^{*\,\Lambda} + B^{*\,\Lambda})(A_\Lambda + B_\Lambda) &= 1\,, \\
4\Re\mathrm{e}[B^{*\,\Lambda}(A_\Lambda + B_\Lambda)] &= 1 - M/r_0\,, \\
\frac{A^{*\,i} + B^{*\,i}}{A^{*\,0} + B^{*\,0}} &= Z^i{}_\infty\,.
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Here $M^2 r_0{}^2 = (M^2 - |\mathcal{Z}_\infty|^2)(M^2 - |\tilde{\mathcal{Z}}_\infty|^2)$, and one can show that the metric is regular in all the $r_0{}^2 > 0$ cases.

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We get

$$\mathcal{H}_\Lambda \overset{M \to |\tilde{\mathcal{Z}}_\infty|}{\longrightarrow} \pm \frac{e^{\mathcal{K}_\infty/2}}{2\sqrt{2}} \left\{ Z^*_{\Lambda\,\infty} - \frac{1}{|\tilde{\mathcal{Z}}_\infty|} \left[-Z^*_{\Lambda\,\infty} \Gamma^{*\,\Sigma}\Gamma_\Sigma + \Gamma_\Lambda Z^{*\,\Sigma}_\infty \Gamma^*_\Sigma \right] \tau \right\} .$$

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   On the event horizon $\tau \to -\infty$ the scalars $Z^i = \mathcal{H}^{*\,i}/\mathcal{H}^{*\,0}$ take the values

$$Z^{*\,i}_{\mathrm{h}} = \frac{\Gamma^i Z^{*\,\Lambda}_\infty \Gamma^*_\Lambda - Z^{*\,i}_\infty \Gamma^{*\,\Sigma}\Gamma_\Sigma}{\Gamma^0 Z^{*\,\Gamma}_\infty \Gamma^*_\Gamma - \Gamma^{*\,\Omega}\Gamma_\Omega},$$

   which depend manifestly on the asymptotic values.

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   which depend manifestly on the asymptotic values.

There is no attractor behavior in a proper sense.

The structure of the extremal non-supersymmetric solution as function of the H^Ms is the same as in the supersymmetric case.

However, no simple *substitution recipe* could have led to it.

Physical properties of the non-extremal solutions

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One can compute the "entropies" of the inner and outer horizons (event horizon (+) and Cauchy horizon) at $\tau \to -\infty$ and $\tau \to +\infty$ resp.:

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They can also be written in the suggestive form

$$S_\pm = \pi \left(\sqrt{N_{\rm R}} \pm \sqrt{N_{\rm L}}\right)^2\,,$$

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The product $S_+ S_-$ is manifestly moduli-independent for all values of r_0:

$$S_+ S_- / \pi^2 = (\Gamma^{*\,\Lambda}\Gamma_\Lambda)^2 \,.$$

The endpoint of the evaporation process of the non-extremal black holes is completely determined by their charges, independently of the moduli $Z^i{}_\infty$:

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➾ Thus, if $\Gamma^{*\,\Lambda}\Gamma_\Lambda > 0$, which is the property that characterizes the supersymmetric attractor , then $|\mathcal{Z}_\infty| > |\tilde{\mathcal{Z}}_\infty|$ and the evaporation process will stop when $M = |\mathcal{Z}_\infty|$ (supersymmetry restoration).

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Or: Where the H^Ms come from

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In the FGK formalism it is always possible to use a different set of variables (Mohaupt & Vaughan arXiv:1112.2876, Meessen, O., Perz & Shahbazi arXiv:1112.3332)

$$U(\tau), Z^i(\tau) \;\; (2n_V+1) \longrightarrow \begin{pmatrix} H^\Lambda \\ H_\Lambda \end{pmatrix} \equiv H^M \, , \;\; (2n_V+2)$$

plus one constraint that will appear automatically.

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We introduce an auxiliary function X and proceed as in the BPS case defining the Kähler-neutral, real, symplectic vectors $\mathcal{R}^M$ and $\mathcal{I}^M$

$$\mathcal{R}^M + i\mathcal{I}^M \equiv \mathcal{V}^\mathcal{M}/X \,.$$

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We know that the components $\mathcal{R}^M$ can be expressed as a function of the $\mathcal{I}^M$s and vice-versa solving the *stabilization equations*.

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Then, we introduce two dual sets of variables

$$\tilde{H}_M \equiv \mathcal{R}_M \,, \qquad H^M \equiv \mathcal{I}^M \,.$$

We define the Hessian potential $\mathsf{W}(H) \equiv \tilde{H}_M(H)H^M$, or $\mathsf{W}(H) \equiv \tilde{H}_M H^M(H)$.

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W is homogenous of second order in the H^M variables and satisfy

$$\partial_M \mathsf{W} \equiv \frac{\partial \mathsf{W}}{\partial H^M} = 2\tilde{H}_M \, , \qquad \partial^M \mathsf{W} \equiv \frac{\partial \mathsf{W}}{\partial \tilde{H}_M} = 2H^M \, ,$$

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Then, the FGK effective action can be written in the form

$$\begin{aligned} I_{\text{eff}}[H] \;\; = \;\; & \int d\tau \left\{ \tfrac{1}{2}\partial_M \partial_N \log \mathsf{W} \left(\dot{H}^M \dot{H}^N + \tfrac{1}{2}\mathcal{Q}^M \mathcal{Q}^N \right) \right. \\ & \left. + \left(\mathsf{W}^{-1} \dot{H}^M H_M \right)^2 + \left(\mathsf{W}^{-1} \mathcal{Q}^M H_M \right)^2 \right\} , \end{aligned}$$

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Having the $H^M(\tau)$ that solve this action, the black-hole solution is given by

$$e^{-2U(\tau)} = \mathsf{W}[H(\tau)] \,, \qquad Z^i(\tau) = \frac{\tilde{H}^i(H) + iH^i}{\tilde{H}^0(H) + iH^0} \,.$$

This shows that we can write all the static black-hole solutions of a given model $N = 2 \;\; d = 4$ supergravity exactly in the same way in terms of the functions $H^M(\tau)$.

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The equations of motion of the H^Ms are

$$-\tfrac{1}{2}\partial_M\partial_N \log \mathsf{W}\left(\dot{H}^M\dot{H}^N - \tfrac{1}{2}\mathcal{Q}^M\mathcal{Q}^N\right) + \left(\mathsf{W}^{-1}\dot{H}^M H_M\right)^2 - (\mathsf{W}^{-1}\mathcal{Q}^M H_M)^2 = r$$

$$\tfrac{1}{2}\partial_M \log \mathsf{W}\left(\ddot{H}^M - r_0{}^2 H^M\right) + \left(\mathsf{W}^{-1}\dot{H}^M H_M\right)^2 = 0$$

$$\tfrac{1}{2}\partial_P\partial_M\partial_N \log \mathsf{W}\left[\dot{H}^M\dot{H}^N - \tfrac{1}{2}\mathcal{Q}^M\mathcal{Q}^N\right] + \partial_P\partial_M \log \mathsf{W}\,\ddot{H}^M - \frac{d}{d\tau}\left(\frac{\partial\Lambda}{\partial\dot{H}^P}\right) + \frac{\partial\Lambda}{\partial H^P} = 0$$

with

$$\Lambda \equiv \left(\mathsf{W}^{-1}\dot{H}^M H_M\right)^2 + (\mathsf{W}^{-1}\mathcal{Q}^M H_M)^2\,.$$

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\tfrac{1}{2}\partial_P\partial_M\partial_N \log \mathsf{W}\left[\dot{H}^M\dot{H}^N - \tfrac{1}{2}\mathcal{Q}^M\mathcal{Q}^N\right] + \partial_P\partial_M \log \mathsf{W}\,\ddot{H}^M - \frac{d}{d\tau}\left(\frac{\partial\Lambda}{\partial\dot{H}^P}\right) + \frac{\partial\Lambda}{\partial H^P} &= 0
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In the extremal case $r_0 = 0$ one sees immediately that $\dot{H}^P = \pm\frac{1}{\sqrt{2}}\mathcal{Q}^P$ satisfying the no-NUT condition $\dot{H}^P H_P = 0$ solve all the equations.

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$$-\tfrac{1}{2}\partial_M \partial_N \log \mathsf{W} \left(\dot{H}^M \dot{H}^N - \tfrac{1}{2}\mathcal{Q}^M \mathcal{Q}^N\right) + \left(\mathsf{W}^{-1}\dot{H}^M H_M\right)^2 - \left(\mathsf{W}^{-1}\mathcal{Q}^M H_M\right)^2 \;=\; r$$

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In the extremal case $r_0 = 0$ one sees immediately that $\dot{H}^P = \pm\frac{1}{\sqrt{2}}\mathcal{Q}^P$ satisfying the no-NUT condition $\dot{H}^P H_P = 0$ solve all the equations. We are searching for more general results.

8 – FGK formalism for $d \geq 4$ and $p \geq 0$

(de Antonio, O., Shahbazi, to appear.)

The FGK formalism can be generalized to higher numbers of spacetime (d) and worldvolume (p) dimensions for generic actions including $(p+1)$-form potentials $A^{\Lambda}_{(p+1)}$

$$
\begin{aligned}
I &= \int d^d x \sqrt{|g|}\, \{ R + \mathcal{G}_{ij}(\phi)\partial_\mu \phi^i \partial^\mu \phi^j \\
&\quad + 4\tfrac{(-1)^p}{(p+2)!} \left[I_{\Lambda\Sigma}(\phi) F^{\Lambda}_{(p+2)} \cdot F^{\Sigma}_{(p+2)} + \xi^2 R_{\Lambda\Sigma}(\phi) F^{\Lambda}_{(p+2)} \star F^{\Sigma}_{(p+2)} \right] \Big\} \,,
\end{aligned}
$$

where the last term occurs only when $p = \tilde{p} = (d-4)/2$ and

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R_{\Lambda\Sigma}(\phi) = -\xi^2 R_{\Sigma\Lambda}(\phi)\,, \qquad \xi^2 = (-1)^{\frac{d}{2}+1} = (-1)^{p+1}\,.
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$$R_{\Lambda\Sigma}(\phi) = -\xi^2 R_{\Sigma\Lambda}(\phi)\,, \qquad \xi^2 = (-1)^{\frac{d}{2}+1} = (-1)^{p+1}\,.$$

We need a generalization of the ansatz that can describe single, static, regular, black p-branes with flat worldvolume in the directions $\vec{y}_{(p)} = (y_1, \cdots, y_p)$ living in a spacetime of $d = p + \tilde{p} + 4$ dimensions.

This ansatz is

$$
\begin{aligned}
ds^2_{(d)} &= e^{\frac{2}{p+1}\tilde{U}}\left[e^{\frac{2p}{p+1}r_0\rho}dt^2 - e^{-\frac{2}{p+1}r_0\rho}d\vec{y}^{\,2}_{(p)}\right] - e^{-\frac{2}{\tilde{p}+1}\tilde{U}}\gamma_{(\tilde{p}+3)\,mn}dx^m dx^n\,, \\
\gamma_{(\tilde{p}+3)\,mn}dx^m dx^n &\equiv \left[\frac{r_0}{\sinh(r_0\rho)}\right]^{\frac{2}{\tilde{p}+1}}\left[\left(\frac{r_0}{\sinh(r_0\rho)}\right)^2\frac{d\rho^2}{(\tilde{p}+1)^2} + d\Omega^2_{(\tilde{p}+2)}\right],
\end{aligned}
$$

and only contains an independent function $\tilde{U}$ and the extremality parameter r_0.

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$$
\begin{aligned}
ds^2_{(d)} &= e^{\frac{2}{p+1}\tilde{U}}\left[e^{\frac{2p}{p+1}r_0\rho}dt^2 - e^{-\frac{2}{p+1}r_0\rho}d\vec{y}^{\,2}_{(p)}\right] - e^{-\frac{2}{\tilde{p}+1}\tilde{U}}\gamma_{(\tilde{p}+3)\,mn}dx^m dx^n\,, \\
\gamma_{(\tilde{p}+3)\,mn}dx^m dx^n &\equiv \left[\frac{r_0}{\sinh(r_0\rho)}\right]^{\frac{2}{\tilde{p}+1}}\left[\left(\frac{r_0}{\sinh(r_0\rho)}\right)^2\frac{d\rho^2}{(\tilde{p}+1)^2} + d\Omega^2_{(\tilde{p}+2)}\right],
\end{aligned}
$$

and only contains an independent function $\tilde{U}$ and the extremality parameter r_0.

The effective action is

$$
I_{\rm eff}[\tilde{U},\phi^i] = \int d\tau\left\{(\dot{\tilde{U}})^2 + \tfrac{(p+1)(\tilde{p}+2)}{d-2}\mathcal{G}_{ij}\dot{\phi}^i\dot{\phi}^j - e^{2\tilde{U}}V_{\rm bb} + {r_0}^2\right\},
$$

where we have defined the black-brane potential

$$
-V_{\rm bb}(\phi,Q) \equiv -\tfrac{1}{2}Q^M Q^N \mathcal{M}_{MN}\,, \quad (\mathcal{M}_{MN}) \equiv \begin{pmatrix} (I-\xi^2 R I^{-1}R)_{\Lambda\Sigma} & \xi^2(RI^{-1})_{\Lambda}{}^{\Sigma} \\ -(I^{-1}R)^{\Lambda}{}_{\Sigma} & (I^{-1})^{\Lambda\Sigma} \end{pmatrix},
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$\mathcal{M}_{MN}$ is an $O(n,n)$ (resp. $Sp(n,n)$) matrix when $\xi^2 = +1$ (resp. -1).

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The non-extremal black-brane solutions seem to have the similar features for all p. An H-FGK formalism exists for theories associated to certain supergravity theories: black holes and black strings in $N = 2$, d=5 supergravity (Mohaupt & Waite `arXiv:0906.3451`, Mohaupt & Vaughan `arXiv:1006.3439` & `arXiv:1112.2876`, Meessen, O., Perz & Shahbazi `arXiv:1112.3332` and work to appear).

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  We have proven that part of our ansatz is completely general, constructing a formalism ("H-FGK") that simplifies the construction of extremal and non-extremal black hole solutions.
- ★ We have extended the FGK formalism to higher spacetime and worldvolume dimensions and the H-FGK formalism for other cases, which will allow us to find new solutions and new general results.

We are closer to determining the general form of all single, static, black-hole and black-string solutions of $N = 2$,$d = 4, 5$ theories.