Supersymmetric solutions and attractor equations in 4-dimensional supergravities

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Talk given on the 20th of November 2010 at the *School and Workshop on D-brane Instantons, Wall Crossing and Microstate Counting, ICTP, Trieste*

Plan of the Talk:

1 – The search for all 4-d susy solutions

[a] Tod (1983) (Pure $N = 2$ $d = 4$). Meessen & O. ($N = 2$ $d = 4$ with vector multiplets); Hübscher, Meessen & O. ($N = 2$ $d = 4$ with vector multiplets and hypermultiplets).

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Spinor-bilinears method

The spinor -bilinears method is specially suited for the $N = 2, d = 4$ case[a]: a Killing spinor consists of 2 Weyl spinors that can be used to construct a tetrad and the complete geometry (equivalent to the Newman -Penrose formalism).

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> For $N > 2$ there are too many spinor bilinears and we do not know how to extract the (**not** spacetime-geometric) information they must surely contain wothout breaking the symmetries of the theory.

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In this talk we are going to show how to solve those problems and determine the general form of **all** the timelike supersymmetric solutions of all $d = 4$ supergravities using the **spinor-bilinear method**.

One of our main results is that the timelike supersymmetric solutions of $N > 2, d = 4$ theories are related to those of the $N = 2, d = 4$ theories found in Hübscher, Meessen & O. (2006).

We start by reviewing them.

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The $N = 2$ supergravity multiplet is

$$\left\{e^{a}{}_{\mu}, \psi_{I\,\mu}, A^{IJ}{}_{\mu}\right\}\,, \quad I, J, \cdots = 1, 2\,, \quad \Rightarrow A^{IJ}{}_{\mu} = A^{0}{}_{\mu}\varepsilon^{IJ}\,.$$

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The (n) $N = 2$ vector multiplets are

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This is a extremely redundant (but useful) description of the scalars .

The supersymmetry transformations of the fermions are

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\delta_\epsilon \psi_{I\,\mu} &= \mathfrak{D}_\mu \epsilon_I + \varepsilon_{IJ}\, T^+{}_{\mu\nu}\gamma^\nu\, \epsilon^J\,, \\
\delta_\epsilon \lambda^{iI} &= i\, \not{\partial} Z^i \epsilon^I \;+\; \varepsilon^{IJ}\, \not{G}^{i\,+}\, \epsilon_J\,. \\
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where the graviphoton and matter vector field strengths are

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and where $\mathsf{U}^{\alpha I}{}_u(q)$ is the *Quadbein*. The action for the bosonic fields is

$$\begin{aligned}S \;\; = \;\; & \int d^4x\sqrt{|g|}\,\big[R + 2\mathcal{G}_{ij^*}\partial_\mu Z^i \partial^\mu Z^{*j^*} + 2\mathsf{H}_{uv}\partial_\mu q^u \partial^\mu q^v \\ & +2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda\,\mu\nu} F^\Sigma{}_{\mu\nu} - 2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda\,\mu\nu} \star F^\Sigma{}_{\mu\nu}\big]\,.\end{aligned}$$

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The goal is to find **all** the bosonic field configurations $\{e^a{}_\mu, A^\Lambda{}_\mu, Z^i, q^u\}$ such that the above KSEs admit at least one solution ϵ^I.

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5. Impose the independent equations of motion on the supersymmetric configurations we just identified.

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The 4-d Fierz identities imply that $V_a \equiv V^I{}_{I\,a}$ is always non-spacelike:

$$V^2 = -V^I{}_J \cdot V^J{}_I = 2M^{IJ} M_{IJ} = 4|X|^2 \geq 0\,.$$

We only consider the timelike case $X \neq 0$ in which all $V^I{}_{J\,a}$ are independent.

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with $\sigma^0 = 1$ and σ^m the 2×2 Pauli matrices as an orthonormal tetrad in which $V^0 = \sqrt{2} V$ is timelike and the V^ms are spacelike. (**This will not work for $N > 2$!**)

5 – The $N = 2$ Killing Spinor Identities (KSI)s

If we assume that a given bosonic field configuration admits a Killing spinor ϵ_I, then we find that the (*off-shell*) "equations of motion" $\{\mathcal{E}^{\mu\nu}, \mathcal{E}^{\mu}, \mathcal{E}^{i}, \mathcal{E}_{u}\}$ satisfy the KSIs:

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6. $\mathcal{E}_{i^*} = 2\left(\frac{X}{X^*}\right)^{1/2} \langle\, \mathcal{E}^{0} \mid \mathcal{D}_{i^*}\mathcal{V}^*\,\rangle$, ($\Rightarrow$ attractor mechanism)

The only independent equations of motion that have to be imposed on $N = 2$, $d = 4$ supersymmetric configurations are

$$\mathcal{E}^0 = 0\,.$$

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($\Rightarrow$ No Kähler nor $SU(2)$ gauge -fixing are necessary!)

2. The components of $\mathcal{I}$ are given by a symplectic vector real functions $\mathcal{H}$ harmonic in the 3-dimensional transverse space with metric $\gamma_{\underline{mn}}$:

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$$Z^i = \frac{\mathcal{V}^i/X}{\mathcal{V}^0/X} = \frac{\mathcal{R}^i + i\mathcal{I}^i}{\mathcal{R}^0 + i\mathcal{I}^0}\,.$$

5. The hyperscalars $q^u(x)$ are the mappings satisfying

$$\mathsf{U}^{\alpha J}{}_m\,(\sigma^m)_J{}^I = 0\,, \qquad \mathsf{U}^{\alpha J}{}_n \equiv V_n{}^{\underline{m}}\partial_{\underline{m}} q^u\,\mathsf{U}^{\alpha J}{}_u\,.$$

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7. The vector field strengths are

$$\mathcal{F} = -\tfrac{1}{2} d(\mathcal{R} \hat{V}) - \tfrac{1}{2} \star (\hat{V} \wedge d\mathcal{I}) \,, \qquad \hat{V} = 2\sqrt{2} |X|^2 (dt + \omega) \,.$$

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All 4-d supergravity multiplets can be written in the form

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The price to pay for using this representation is that all the fields that can be related by $SU(N)$ duality relations, are:

- $N = 4$: $P^{*\,i\,IJ} = \frac{1}{2}\varepsilon^{IJKL} P_{i\,KL}$, and $\lambda_{iI} = \frac{1}{3!}\varepsilon_{IJKL}\lambda_i{}^{IJK}$.
- $N = 6$: $P^{*\,IJ} = \frac{1}{4!}\varepsilon^{IJK_1\cdots K_4} P_{K_1\cdots K_4}$, $\chi_{IJK} = \frac{1}{3!}\varepsilon_{IJKLMN}\lambda^{IJK}$, and $\chi^{I_1\cdots I_5} = \varepsilon^{I_1\cdots I_5 J}\lambda_J$.
- $N = 8$: $P^{*\,I_1\cdots I_4} = \frac{1}{4!}\varepsilon^{I_1\cdots I_4 J_1\cdots J_4} P_{J_1\cdots J_4}$, and $\chi_{I_1 I_2 I_3} = \frac{1}{5!}\varepsilon_{I_1 I_2 I_3 J_1\cdots J_5}\chi^{J_1\cdots J_5}$.

  These constraints must be taken into account in the action.

The scalars are encoded into the $2\bar{n}$-dimensional ($\bar{n} \equiv n + \frac{N(N-1)}{2}$) symplectic vectors

$$\mathcal{V}_{IJ} = \begin{pmatrix} f^{\Lambda}{}_{IJ} \\ h_{\Lambda\, IJ} \end{pmatrix}, \qquad \text{and} \qquad \mathcal{V}_{i} = \begin{pmatrix} f^{\Lambda}{}_{i} \\ h_{\Lambda\, i} \end{pmatrix}, \qquad \Lambda = 1, \cdots, \bar{n}\,,$$

normalized

$$\langle \mathcal{V}_{IJ} \mid \mathcal{V}^{*\,KL} \rangle = -2i\delta^{KL}{}_{IJ}\,, \qquad \langle \mathcal{V}_{i} \mid \mathcal{V}^{*\,j} \rangle = -i\delta_{i}{}^{j}\,.$$

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They can be combined into the $Usp(\bar{n}, \bar{n})$ matrix

$$U \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} f + ih & f^{*} + ih^{*} \\ f - ih & f^{*} - ih^{*} \end{pmatrix}.$$

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The graviphotons $A^{IJ}{}_{\mu}$ do not appear directly, only through the "dressed" vectors

$$A^{\Lambda}{}_{\mu} \equiv \tfrac{1}{2} f^{\Lambda}{}_{IJ} A^{IJ}{}_{\mu} + f^{\Lambda}{}_{i} A^{i}{}_{\mu}\,.$$

The supersymmetry transformations of the fermioninc fields are

$$\begin{aligned}
\delta_\epsilon \psi_{I\mu} &= \mathfrak{D}_\mu \epsilon_I + T_{IJ}{}^{+}{}_{\mu\nu}\gamma^\nu \epsilon^J \,, \\
\delta_\epsilon \chi_{IJK} &= -\tfrac{3i}{2}\, \not{T}_{[IJ}{}^{+}\epsilon_{K]} + i\, \not{P}_{IJKL}\epsilon^L \,, \\
\delta_\epsilon \lambda_{iI} &= -\tfrac{i}{2}\, \not{T}_i{}^{+}\epsilon_I + i\, \not{P}_{iIJ}\epsilon^J \,, \\
\delta_\epsilon \chi_{IJKLM} &= -5i\, \not{P}_{[IJKL}\epsilon_{M]} + \tfrac{i}{2}\varepsilon_{IJKLMN}\, \not{T}^{-}\epsilon^N + \tfrac{i}{4}\varepsilon_{IJKLMNOP}\, \not{T}^{NO-}\epsilon^P \,, \\
\delta_\epsilon \lambda_{iIJK} &= -3i\, \not{P}_{i[IJ}\epsilon_{K]} + \tfrac{i}{2}\varepsilon_{IJKL}\, \not{T}_i{}^{-}\epsilon^L + \tfrac{i}{4}\varepsilon_{IJKLMN}\, \not{T}^{LM-}\epsilon_N \,,
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where the graviphoton and matter vector field strengths are

$$T_{IJ}{}^+ = \langle \mathcal{V}_{IJ} \mid \mathcal{F}^+ \rangle \,, \quad T_i{}^+ = \langle \mathcal{V}_i \mid \mathcal{F}^+ \rangle \,, \quad \mathcal{F}_\Lambda{}^+ = \mathcal{N}^*_{\Lambda\Sigma} F^{\Sigma\,+} \,,$$

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and where

$$\mathfrak{D}_\mu \epsilon_I \equiv \nabla_\mu \epsilon_I - \epsilon_J \Omega_\mu{}^J{}_I \,,$$

and $\Omega_\mu{}^J{}_I$ is the pullback of the connection of the scalar manifold ($\subset U(N)$).

The action for the bosonic fields is

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\begin{aligned}
S \quad = \quad & \int d^4x \sqrt{|g|}\, \Big[R + 2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda\,\mu\nu} F^{\Sigma}{}_{\mu\nu} - 2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda\,\mu\nu} \star F^{\Sigma}{}_{\mu\nu} \\
& \qquad\qquad + \tfrac{2}{4!}\alpha_1 P^{*\,IJKL}{}_{\mu} P_{IJKL}{}^{\mu} + \alpha_2 P^{*\,iIJ}{}_{\mu} P_{iIJ}{}^{\mu} \Big] \,,
\end{aligned}
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where

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\mathcal{N} = hf^{-1} = \mathcal{N}^T \,, \qquad h_\Lambda = \mathcal{N}_{\Lambda\Sigma} f^\Sigma \,. \qquad \mathfrak{D} h_\Lambda = \mathcal{N}^*_{\Lambda\Sigma} \mathfrak{D} f^\Lambda \,.
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The action for the bosonic fields is

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$$\left. + \tfrac{2}{4!}\alpha_1 P^{*\,IJKL}{}_{\mu} P_{IJKL}{}^{\mu} + \alpha_2 P^{*\,iIJ}{}_{\mu} P_{iIJ}{}^{\mu} \right] ,$$

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For $N = 3$ **:** $\mathcal{E}^{iIJ} = \mathfrak{D}^{\mu} P^{*\,iIJ}{}_{\mu} + 2T^{i\,-}{}_{\mu\nu} T^{IJ\,-\,\mu\nu}$.

For $N = 4$ **:**

$$\begin{cases} \mathcal{E}^{IJKL} = \mathfrak{D}^{\mu} P^{*\,IJKL}{}_{\mu} + 6T^{[IJ|-}{}_{\mu\nu} T^{|KL]-\,\mu\nu} \\ \qquad + P^{*\,IJKL\,A} P^{*\,ij}{}_{A} T_{i}{}^{+}{}_{\mu\nu} T_{j}{}^{+\,\mu\nu} , \\ \mathcal{E}^{iIJ} = \mathfrak{D}^{\mu} P^{*\,iIJ}{}_{\mu} + T^{i\,-}{}_{\mu\nu} T^{IJ\,-\,\mu\nu} + \tfrac{1}{2}\varepsilon^{IJKL} T_{i}{}^{+}{}_{\mu\nu} T_{KL}{}^{+\,\mu\nu} . \end{cases}$$

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For $N = 5$ **:** $\mathcal{E}^{IJKL} = \mathfrak{D}^{\mu} P^{*\,IJKL}{}_{\mu} + 6T^{[IJ|-}{}_{\mu\nu} T^{|KL]-\,\mu\nu}$. etc.

8 – The all-N Killing Spinor Equations (KSEs)

For all values of N the independent KSEs take the form

$$
\begin{aligned}
\mathfrak{D}_\mu \epsilon_I + T_{IJ}{}^+{}_{\mu\nu}\gamma^\nu \epsilon^J &= 0\,, \\
\not{P}_{IJKL}\epsilon^L - \tfrac{3}{2}\,\not{T}_{[IJ}{}^+\epsilon_{K]} &= 0\,, \\
\not{P}_{i\,IJ}\epsilon^J - \tfrac{1}{2}\,\not{T}_i{}^+\epsilon_I &= 0\,, \\
\not{P}_{[IJKL}\epsilon_{M]} &= 0\,, \\
\not{P}_{i\,[IJ}\epsilon_{K]} &= 0\,.
\end{aligned}
$$

The last two KSEs should only be considered for $N = 5$ and $N = 3$, resp.

8 – The all-N Killing Spinor Equations (KSEs)

For all values of N the independent KSEs take the form

$$
\begin{aligned}
\mathfrak{D}_\mu \epsilon_I + T_{IJ}{}^{+}{}_{\mu\nu}\gamma^\nu \epsilon^J &= 0\,,\\
\not{P}_{IJKL}\epsilon^L - \tfrac{3}{2}\,\not{T}_{[IJ}{}^{+}\epsilon_{K]} &= 0\,,\\
\not{P}_{i\,IJ}\epsilon^J - \tfrac{1}{2}\,\not{T}_i{}^{+}\epsilon_I &= 0\,,\\
\not{P}_{[IJKL}\epsilon_{M]} &= 0\,,\\
\not{P}_{i\,[IJ}\epsilon_{K]} &= 0\,.
\end{aligned}
$$

The last two KSEs should only be considered for $N=5$ and $N=3$, resp.

Again, our goal is to find **all** the bosonic field configurations $\{e^a{}_\mu, A^\Lambda{}_\mu, P_{IJKL\,\mu}, P_{i\,IJ\,\mu}\}$ such that the above KSEs admit at least one solution ϵ^I.

9 – The all-N spinor-bilinears algebra

The independent bilinears that we can construct with one $U(N)$ vector of Weyl spinors ϵ_I are:

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3. We can choose a tetrad $\{e^a{}_\mu\}$ such that $e^0{}_\mu \equiv \frac{1}{\sqrt{2}}|M|^{-1}V_\mu$. Then, **defining** $V^m{}_\mu \equiv |M|e^m{}_\mu$ we can decompose

$$V^I{}_{J\,\mu} = \tfrac{1}{2}\mathcal{J}^I{}_J V_\mu + \tfrac{1}{\sqrt{2}}(\sigma^m)^I{}_J V^m{}_\mu\,,$$

where $\mathcal{J}^I{}_J = 2M^{IK}M_{JK}|M|^{-2}$ is a rank 2 projector (Tod):

$$\mathcal{J}^2 = \mathcal{J}\,, \qquad \mathcal{J}^I{}_I = +2\,, \qquad \mathcal{J}^I{}_J\epsilon^J = \epsilon^I\,.$$

The main properties satisfied by the three σ^m matrices are:

$$
\begin{aligned}
\sigma^m \sigma^n &= \delta^{mn} \mathcal{J} + i\varepsilon^{mnp} \sigma^p \,, \\
\mathcal{J} \sigma^m &= \sigma^m \mathcal{J} = \sigma^m \,, \\
(\sigma^m)^I{}_I &= 0\,, \\
\mathcal{J}^K{}_J \mathcal{J}^L{}_I &= \tfrac{1}{2} \mathcal{J}^K{}_I \mathcal{J}^L{}_J + \tfrac{1}{2} (\sigma^m)^K{}_I (\sigma^m)^L{}_J \,, \\
M_{K[I} (\sigma^m)^K{}_{J]} &= 0\,, \\
2|M|^{-2} M_{LI} (\sigma^m)^I{}_J M^{JK} &= (\sigma^m)^K{}_L \,,
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\end{aligned}
$$

$\{\mathcal{J}, \sigma^1, \sigma^2, \sigma^3\}$ is an x-dependent basis of a $\mathfrak{u}(2)$ subalgebra of $\mathfrak{u}(N)$ in the 2-dimensional eigenspace of $\mathcal{J}$ of eigenvalue $+1$ and provide a basis in the space of Hermitean matrices A satisfying $\mathcal{J} A \mathcal{J} = A$

10 – The all-N Killing Spinor Identities (KSIs)

If we assume that a given bosonic field configuration admits a Killing spinor ϵ_I, then we find that the (*off-shell*) "equations of motion" $\{\mathcal{E}^{\mu\nu}, \mathcal{E}^{\mu}, \mathcal{E}^{IJKL}, \mathcal{E}^{i\,IJ}\}$ satisfy the KSIs ($\tilde{\mathcal{J}}^{I}{}_{J} \equiv \delta^{I}{}_{J} - \mathcal{J}^{I}{}_{J}$):

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3. $$\left\{ \begin{array}{rcl} \mathcal{E}^{MNPQ}\mathcal{J}^{[I}{}_M\tilde{\mathcal{J}}^J{}_N\tilde{\mathcal{J}}^K{}_P\tilde{\mathcal{J}}^{L]}{}_Q & = & 0\,, \\ \mathcal{E}^{i\,MN}\mathcal{J}^{[I}{}_M\tilde{\mathcal{J}}^{J]}{}_N & = & 0\,, \end{array} \right.$$ ($\Rightarrow$ no attractor mechanism)

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$N = 3$ **:** $\mathcal{E}^{i\,IJ} = -2\sqrt{2}\frac{M^{IJ}}{|M|}\langle\,\mathcal{E}^0 \mid \mathcal{V}^{*\,i}\,\rangle\,,$

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$$N=4:\ \left\{\begin{array}{rcl} \mathcal{E}^{IJKL} & = & -2\sqrt{2}\frac{M^{[IJ|}}{|M|}\langle \mathcal{E}^0 \mid \mathcal{V}^{*\,|KL]}\rangle\,, \\ \\ \mathcal{E}_{iIJ} & = & -2\sqrt{2}\left\{\frac{M_{IJ}}{|M|}\langle \mathcal{E}^0 \mid \mathcal{V}_i\rangle + \tfrac{1}{2}\varepsilon_{IJKL}\frac{M^{KL}}{|M|}\langle \mathcal{E}^0 \mid \mathcal{V}^{*\,i}\rangle\right\}\,, \end{array}\right.$$

etc.

The only independent equations of motion that have to be imposed on **any** $d = 4$ supersymmetric configuration are

$$\mathcal{E}^0 = 0\,.$$

11 – The all-N supersymmetric solutions

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   We also have to impose the constraint

$$\mathcal{J} d\sigma^m \mathcal{J} = 0\,.$$

Once the $U(2)$ subgroup has been chosen, we can split the Vielbeins $P_{IJKL\,\mu}$ and $P_{i\,IJ\,\mu}$, into associated to the would-be vector multiplets in the $N=2$ truncation

$$P_{IJKL}\,\mathcal{J}^{I}{}_{[M}\mathcal{J}^{J}{}_{N}\tilde{\mathcal{J}}^{K}{}_{P}\tilde{\mathcal{J}}^{L}{}_{Q]}\,,\quad \text{and}\quad P_{i\,IJ}\,\mathcal{J}^{I}{}_{[K}\mathcal{J}^{J}{}_{L]}\,,$$

which are driven by the *attractor mechanism* (*i.e.* they are determined by the electric and magnetic charges) and those associated to the hypermultiplets

$$P_{IJKL}\,\mathcal{J}^{I}{}_{[M}\tilde{\mathcal{J}}^{J}{}_{N}\tilde{\mathcal{J}}^{K}{}_{P}\tilde{\mathcal{J}}^{L}{}_{Q]}\,,\quad \text{and}\quad P_{i\,IJ}\,\mathcal{J}^{I}{}_{[K}\tilde{\mathcal{J}}^{J}{}_{L]}\,.$$

which are not.

In hyper-less solutions (*e.g.* black holes) the σ^ms matrices are not needed at all.

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where

$$\begin{aligned} |M|^{-2} &= (M^{IJ}M_{IJ})^{-2} = \langle\, \mathcal{R} \mid \mathcal{I}\,\rangle\,, \\ (d\omega)_{mn} &= 2\epsilon_{mnp}\langle\, \mathcal{I} \mid \partial^p\mathcal{I}\,\rangle\,. \end{aligned}$$

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Its spin connection $\varpi^{mn}$is related to Ω, by

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6. The scalars in the vector multiplets in the associated $N = 2$ truncation

$$P_{IJKL}\,\mathcal{J}^I{}_{[M}\mathcal{J}^J{}_N\tilde{\mathcal{J}}^K{}_P\tilde{\mathcal{J}}^L{}_{Q]}\,, \quad \text{and} \quad P_{i\,IJ}\,\mathcal{J}^I{}_{[K}\mathcal{J}^J{}_{L]}\,,$$

can be found from $\mathcal{R}$ and $\mathcal{I}$, while those in the hypers must be found independently by solving

$$P_{IJKL\,m}\,\mathcal{J}^I{}_{[M}\tilde{\mathcal{J}}^J{}_N\tilde{\mathcal{J}}^K{}_P\tilde{\mathcal{J}}^L{}_{Q]}(\sigma^m)^Q{}_R = 0\,,$$

$$P_{i\,IJ\,m}\,\mathcal{J}^I{}_{[K}\tilde{\mathcal{J}}^J{}_{L]}(\sigma^m)^L{}_M = 0\,,$$

which solve their equations of motion according to the *Killing Spinor Identities*.

Attractor flow equations

A simple derivation of the attractor flow eqs. in $N = 1, d = 5$ supergravity

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The matter vector fields $A^{x}{}_{\mu}$ and the graviphoton $A^{0}{}_{\mu}$ are combined into an $SO(n+1)$ vector $A^{I}{}_{\mu}$ with $I = 0, x$. To make manifest the symmetries, the n real scalars ϕ^{x} are described by $n+1$ functions $h^{I}(\phi)$ which are constrained:

$$C_{IJK}h^{I}h^{J}h^{K} = 1\,.$$

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$$\{A^{x}{}_{\mu}, \lambda^{ix}, \phi^{x}\}\,, \qquad x = 1, \cdots, n\,.$$

The matter vector fields $A^{x}{}_{\mu}$ and the graviphoton $A^{0}{}_{\mu}$ are combined into an $SO(n+1)$ vector $A^{I}{}_{\mu}$ with $I = 0, x$. To make manifest the symmetries, the n real scalars ϕ^{x} are described by $n+1$ functions $h^{I}(\phi)$ which are constrained:

$$C_{IJK} h^{I} h^{J} h^{K} = 1\,.$$

We introduce a function f and assume ($h_{I} \equiv C_{IJK} h^{J} h^{K}$)

$$h_{I}/f \equiv l_{I} + q_{I}\rho\,,$$

for some coordinate ρ.

Attractor flow equations

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We introduce a function f and assume ($h_I \equiv C_{IJK}h^J h^K$)

$$h_I/f \equiv l_I + q_I\rho\,,$$

for some coordinate ρ. Let's define the central charge

$$\mathcal{Z}[\phi(\rho), q] \equiv h^I(\phi) q_I\,.$$

Then, using $h^I h_I = 1$ and $dh^I h_I = h^I dh_I = 0$

$$df^{-1} = d(h^I h_I / f) = h^I d(h_I / f)\,,$$

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Using now the above properties plus $h^I{}_x h_{Iy} = g_{xy}$, where $h_{Iy} = -\sqrt{3}\partial_y h_I$ and $h^I{}_x = \sqrt{3}\partial_x h_I$

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from which we get

$$\frac{d\phi^x}{d\rho} = -f g^{xy} \partial_y \mathcal{Z}[\phi(\rho), q]\,.$$

The system of ordinary differential equations

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At the attractor point $\rho_{\rm attract}$ $\phi(\rho_{\rm attract}) = \phi_{\rm fix}$

$$\frac{df^{-1}}{d\rho}\Big|_{\rho=\rho_{\rm attract}} = \mathcal{Z}[\phi_{\rm fix}(q), q] \equiv \mathcal{Z}_{\rm fix}(q)\,.$$

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We define the central charges

$$\begin{aligned}
\mathcal{Z}_{IJ}[\phi(\rho), q] &\equiv \langle\, \mathcal{V}_{IJ} \mid q\,\rangle = p^{\Lambda}h_{\Lambda\, IJ} - q_{\Lambda}f^{\Lambda}{}_{IJ}\,, \\
\mathcal{Z}_{i}[\phi(\rho), q] &\equiv \langle\, \mathcal{V}_{i} \mid q\,\rangle = p^{\Lambda}h_{\Lambda\, i} - q_{\Lambda}f^{\Lambda}{}_{i}\,.
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Then

$$\begin{aligned}
\mathfrak{D}\frac{M^{IJ}}{|M|^2} &= \mathfrak{D}\left(\frac{M^{KL}}{|M|^2}\tfrac{i}{2}\langle\, \mathcal{V}_{KL} \mid \mathcal{V}^{*\,IJ} \,\rangle\right) = \tfrac{i}{2}\mathfrak{D}\langle\,(\mathcal{R} + i\mathcal{I}) \mid \mathcal{V}^{*\,IJ}\,\rangle \\
&= \tfrac{i}{2}\langle\, d(\mathcal{R} + i\mathcal{I}) \mid \mathcal{V}^{*\,IJ}\,\rangle = \tfrac{i}{2}\langle\, d(\mathcal{R} - i\mathcal{I}) \mid \mathcal{V}^{*\,IJ}\,\rangle - \langle\, d\mathcal{I} \mid \mathcal{V}^{*\,IJ}\,\rangle \\
&= \tfrac{i}{2}\frac{M_{KL}}{|M|^2}\langle\, d\mathcal{V}^{*\,KL} \mid \mathcal{V}^{*\,IJ}\,\rangle - \langle\, q \mid \mathcal{V}^{*\,IJ}\,\rangle d\rho \\
&= \tfrac{1}{2}P^{*\,KLIJ}\frac{M_{KL}}{|M|^2} + \mathcal{Z}^{*\,IJ}[\phi(\rho), q]d\rho\,.
\end{aligned}$$

With the above identitiy we can compute

$$d|M|^{-2} = \frac{M_{IJ}}{|M|^2}\mathfrak{D}\frac{M^{IJ}}{|M|^2} + \frac{M^{IJ}}{|M|^2}\mathfrak{D}\frac{M_{IJ}}{|M|^2} = \frac{M_{IJ}\mathcal{Z}^{*\,IJ} + M^{IJ}\mathcal{Z}_{IJ}}{|M|^2}d\rho\,,$$

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which leads to the flow equation (for all $N \geq 2$)

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We can also compute

$$0 = M^{[IJ}\mathfrak{D}\frac{M^{KL]}}{|M|^2} = M^{[IJ}\mathcal{Z}^{*\,KL]}d\rho + \tfrac{1}{2}P^{*\,MN[IJ}\mathcal{J}^{K}{}_{M}\mathcal{J}^{L]}{}_{N}\,,$$

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which leads to the flow equation ($N \geq 4$)

$$P^{*\,MN[IJ}\mathcal{J}^{K}{}_{M}\mathcal{J}^{L]}{}_{N} = -M^{[IJ}\mathcal{Z}^{*\,KL]}d\rho\,.$$

The third flow equation ($N = 2, 3, 4, 6$) follows from

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\begin{aligned}
\tfrac{1}{2}\frac{M^{IJ}}{|M|^2}P_{iIJ} &= -\tfrac{i}{2}\frac{M^{IJ}}{|M|^2}\langle\, d\mathcal{V}_{IJ} \mid \mathcal{V}_i\,\rangle = -\tfrac{i}{2}\langle\, d(\mathcal{R} + i\mathcal{I}) \mid \mathcal{V}_i\,\rangle \\
&= \langle\, d\mathcal{I} \mid \mathcal{V}_i\,\rangle - \tfrac{i}{2}\langle\, d(\mathcal{R} - i\mathcal{I}) \mid \mathcal{V}_i\,\rangle \\
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and takes the final form

$$P_{i\,KL}\mathcal{J}^{K}{}_{I}\mathcal{J}^{L}{}_{J} = -2M_{IJ}\mathcal{Z}_i d\rho\,.$$

Summarizing, we have obtained 3 differential equations:

$$\begin{cases} \frac{d}{d\rho}|M|^{-1} & = & \Re\mathrm{e}\left(\frac{M^{IJ}\mathcal{Z}_{IJ}}{|M|}\right)\,, \\ P^{*\,MN[IJ}\mathcal{J}^{K}{}_{M}\mathcal{J}^{L]}{}_{N} & = & -M^{[IJ}\mathcal{Z}^{*\,KL]}d\rho\,, \\ P_{i\,KL}\mathcal{J}^{K}{}_{I}\mathcal{J}^{L}{}_{J} & = & -2M_{IJ}\mathcal{Z}_{i}d\rho\,, \end{cases}$$

which govern the metric function $g_{tt} = |M|^2$ and the scalars that would fit into $N = 2$ vector supermultiplets. The supersymmetric attractors are at the solutions of

$$M^{[IJ}\mathcal{Z}^{*\,KL]} = 0\,, \quad \text{(for scalars in the supergravity multiplet)}$$

$$\mathcal{Z}_i = 0\,, \quad \text{(for scalars in vector multiplets as in } N=2\text{)}$$

Only in the $N = 2$ case the last equation is a differential equation.

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- ★ "1-line" derivations of the attractor flow equations can be readily found.

- ★ Much work remains to be done in order to make explicit the construction of the solutions. In particular one has to find general parametrizations of the matrices M^{IJ} and $\mathcal{J}^I{}_J$, solve the *stabilization equations*, impose the covariant constancy of $\mathcal{J}$ etc. (Meessen & O., work in progress).