Supersymmetric solutions of 4-dimensional supergravities

Tomás Ortín

(I.F.T. UAM/CSIC, Madrid)

Based on arXiv:1006.0239. Work done with *P. Meessen* (University of Oviedo) and *S. Vaulà* (IFT UAM/CSIC, Madrid)

Talk given on the 23rd of July 2010 at the *4th Mexican Meeting in Mathematical and Experimental Physics*

Plan of the Talk:

1 – Intro: rôle and importance of supersymmetric solutions

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- ➾ Required for consistency of gravity/gauge (AdS/CFT) correspondence.

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  The (unbroken) supersymmetry of the classical solution plays a crucial role in this and many other problems. This is what makes supersymmetric (BPS) solutions interesting. Many interesting GR solutions **are** supersymmetric .

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This generalizes the concept of isometry, an infinitesimal g.c.t. generated by $\xi^\mu(x)$ that leaves the metric $g_{\mu\nu}$ invariant because it satisfies

$$\delta_\xi g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)} = 0 \,. \qquad \textit{Killing}\,(\textit{Vector})\,\textit{Equation}$$

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Every supersymmetric field configuration has a supersymmetry superalgebra. For instance, the superalgebra of Minkowski spacetime is the Poincaré superalgebra with

$$\{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\} = (\gamma^{\mu}\mathcal{C})_{\alpha\beta} P_\mu \,.$$

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- → In supersymmetric black-hole solutions there is an *attractor mechanism* at work which suppresses primary scalar hair and hints at a microscopic interpretation of the entropy (Ferrara, Kallosh & Strominger, `hep-th/9508072,9602111, 9602136`).

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and we get for any field configuration ϕ^b and any ϵ

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By definition, for supersymmetric ϕ^b we have $\delta_\epsilon \phi^f\big|_{\phi^f=0}$ (ϵ is a Killing spinor) and we obtain the Killing Spinor Identities

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Finally, they provide powerful consistency checks when we try to find large families of supersymmetric solutions, as we are going to do.

The Attractor Mechanism

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It can be shown that at the event horizon $r = r_H$ the scalars ϕ^i and the metric function $r^2 g_{rr}$ take their attractor value which only depends on the conserved charges q_{Λ}, p^{Λ} and not on $\phi^{i}{}_{\infty}$):

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This proves that, at least for these supersymmetric black holes, the Bekenstein -Hawking entropy $S(q,p)$ only depends on charges which are going to be quantized, and therefore it is just a function of integer numbers amenable to a microscopic interpretation.

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- ☞ **2004**: Cariglia & Mac Conamhna ($N = 1$ $d = 7$ and gauged $N = (2, 0)$ $d = 6$)

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- **2003**: Gauntlett & Pakis + Gauntlett, Gutowski & Pakis ($N = 1$ $d = 11$); Gauntlett & Gutowski (Gauged $N = 1$ $d = 5$); Caldarelli & Klemm (Pure gauged $N = 2$ $d = 4$); Gutowski, Martelli & Reall; Chamseddine, Figueroa-O'Farrill & Sabra ($N = (2, 0)$ $d = 6$)
- **2004**: Cariglia & Mac Conamhna ($N = 1$ $d = 7$ and gauged $N = (2, 0)$ $d = 6$)
- **2005**: Bellorín & O. (Pure $N = 4$ $d = 4$ revisited)

2 – The search for all 4-d susy solutions

- Gibbons & Hull (1982) (Pure $N = 2$ supergravity).
- Tod (1983) (Pure $N = 2$ supergravity). $\Rightarrow$ A complete answer is possible.
- Tod (1995) (Pure $N = 4$ supergravity).
- Gauntlett, Gutowski, Hull, Pakis & Reall (2002) (Pure $N = 1$ $d = 5$ supergravity).

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- **2006**: Bellorín, Meessen & O. ($N = 1$ $d = 5$ with vector multiplets); Meessen & O. ($N = 2$ $d = 4$ with vector multiplets); Hübscher, Meessen & O. ($N = 2$ $d = 4$ with vector multiplets and hypermultiplets).

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> For $N > 2$ there are too many spinor bilinears and we do not know how to extract the (**not** spacetime-geometric) information they must surely contain.

In this talk we are going to show how to solve those problems and determine the form of **all** the timelike supersymmetric solutions of all $d = 4$ supergravities using the **spinor-bilinear method**.

3 – Review of the N=2 case

Since the timelike supersymmetric solutions of $N > 2$ turn out to be related to those of $N = 2$ theories (Hübscher, Meessen & O. (2006)), we briefly review them first.

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The $N=2$ supergravity multiplet is

$$\left\{e^{a}{}_{\mu}, \psi_{I\,\mu}, A^{IJ}{}_{\mu}\right\}\,, \quad I,J,\cdots = 1,2\,, \quad \Rightarrow A^{IJ}{}_{\mu} = A^{0}{}_{\mu}\varepsilon^{IJ}\,.$$

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This is a extremely redundant (but useful) description of the scalars .

The supersymmetry transformations of the fermions are

$$\delta_\epsilon \psi_{I\,\mu} = \mathfrak{D}_\mu \epsilon_I + \varepsilon_{IJ}\, T^+{}_{\mu\nu}\gamma^\nu\, \epsilon^J\,,$$

$$\delta_\epsilon \lambda^{iI} = i\, \not{\partial} Z^i \epsilon^I \;+\; \varepsilon^{IJ}\, \not{G}^{i\,+}\, \epsilon_J\,.$$

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where the graviphoton and matter vector field strengths are

$$T^{+} = \langle\, \mathcal{V}\mid \mathcal{F}^{+}\,\rangle\,, \qquad G^{i\,+} = \tfrac{i}{2}\mathcal{G}^{ij^*}\langle\, \mathcal{D}_{j^*}\mathcal{V}^*\mid \mathcal{F}^{+}\,\rangle\,, \qquad \mathcal{F}^{+} \equiv \begin{pmatrix} F^{\Lambda\,+} \\ \mathcal{N}^*_{\Lambda\Sigma}F^{\Sigma\,+} \end{pmatrix}\,,$$

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$\mathfrak{D}$ is the Lorentz-, Kähler- and $SU(2)$- covariant derivative (Kähler + $SU(2) = U(2)$)

$$\mathfrak{D}_\mu \epsilon_I = (\partial_\mu + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} \;+\; \tfrac{i}{2}\, \mathcal{Q}_\mu)\, \epsilon_I \;+\; \mathsf{A}_{\mu\, I}{}^J\, \epsilon_J\,,$$

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and where $\mathsf{U}^{\alpha I}{}_u(q)$ is the *Quadbein*. The action for the bosonic fields is

$$\begin{aligned}
S &= \int d^4x \sqrt{|g|}\, \big[R + 2\mathcal{G}_{ij^*}\partial_\mu Z^i \partial^\mu Z^{*j^*} + 2\mathsf{H}_{uv}\partial_\mu q^u \partial^\mu q^v \\
&\qquad + 2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}F^\Sigma{}_{\mu\nu} - 2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu} \star F^\Sigma{}_{\mu\nu}\big]\,.
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The goal is to find **all** the bosonic field configurations $\{e^{a}{}_{\mu}, A^{\Lambda}{}_{\mu}, Z^{i}, q^{u}\}$ such that the above KSEs admit at least one solution ϵ^{I}.

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5. Impose the independent equations of motion on the supersymmetric configurations we just identified.

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The 4-d Fierz identities imply that $V_a \equiv V^I{}_{I\,a}$ is always non-spacelike:

$$V^2 = -V^I{}_J \cdot V^J{}_I = 2M^{IJ} M_{IJ} = 4|X|^2 \geq 0\,.$$

We only consider the timelike case $X \neq 0$ in which all $V^I{}_{J\,a}$ are independent.

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With them one can construct a tetrad

$$V^a{}_\mu \equiv \tfrac{1}{\sqrt{2}} V^I{}_{J\,\mu} (\sigma^a)^J{}_I\,, \qquad V^I{}_{J\,\mu} = \tfrac{1}{\sqrt{2}} V^a{}_\mu (\sigma^a)^I{}_J\,,$$

with $\sigma^0 = 1$ and σ^m the 2×2 Pauli matrices as an orthonormal tetrad in which $V^0 = \sqrt{2} V$ is timelike and the V^ms are spacelike.

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with $\sigma^0 = 1$ and σ^m the 2×2 Pauli matrices as an orthonormal tetrad in which $V^0 = \sqrt{2} V$ is timelike and the V^ms are spacelike. (**This will not work for $N > 2$!**)

6 – The $N = 2$ Killing Spinor Identities (KSI)s

If we assume that a given bosonic field configuration admits a Killing spinor ϵ_I, then we find that the (*off-shell*) "equations of motion" $\{\mathcal{E}^{\mu\nu}, \mathcal{E}^{\mu}, \mathcal{E}^{i}, \mathcal{E}_{u}\}$ satisfy the KSIs:

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5. $0 = \langle\, \mathcal{E}^{0} \mid \Im\mathrm{m}(\mathcal{V}/X)\,\rangle$, ($\Rightarrow$ no NUT charges) (Bellorín, Meessen, Ortín (2008)).

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5. $0 = \langle\, \mathcal{E}^0 \mid \Im\mathrm{m}(\mathcal{V}/X)\,\rangle$, ($\Rightarrow$ no NUT charges) (Bellorín, Meessen, Ortín (2008)).

6. $\mathcal{E}_{i^*} = 2\left(\frac{X}{X^*}\right)^{1/2}\langle\, \mathcal{E}^0 \mid \mathcal{D}_{i^*}\mathcal{V}^*\,\rangle$, ($\Rightarrow$ attractor mechanism)

The only independent equations of motion that have to be imposed on $N=2$, $d=4$ supersymmetric configurations are

$$\mathcal{E}^0 = 0\,.$$

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4. The scalars Z^i are given by the quotients

$$Z^i = \frac{\mathcal{V}^i/X}{\mathcal{V}^0/X} = \frac{\mathcal{R}^i + i\mathcal{I}^i}{\mathcal{R}^0 + i\mathcal{I}^0}\,.$$

5. The hyperscalars $q^u(x)$ are the mappings satisfying

$$\mathsf{U}^{\alpha J}{}_m \, (\sigma^m)_J{}^I \;=\; 0\,, \qquad\qquad \mathsf{U}^{\alpha J}{}_n \;\equiv\; V_n{}^{\underline{m}} \partial_{\underline{m}} q^u \, \mathsf{U}^{\alpha J}{}_u\,.$$

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$\gamma_{\underline{mn}}$ is determined indirectly from the hyperscalars : its spin connection ϖ^{mn} in the basis $\{V^m\}$ is related to the pullback of the $SU(2)$ connection of the hyper-Kähler manifold $\mathsf{A}^I{}_{J\mu} = \frac{1}{\sqrt{2}}\mathsf{A}^m{}_u(\sigma^m)^I{}_J\partial_\mu q^u$, by

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7. The vector field strengths are

$$\mathcal{F} = -\tfrac{1}{2}d(\mathcal{R}\hat{V}) - \tfrac{1}{2}\star(\hat{V}\wedge d\mathcal{I})\,, \qquad\qquad \hat{V} = 2\sqrt{2}|X|^2(dt+\omega)\,.$$

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All 4-d supergravity multiplets can be written in the form

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The price to pay for using this representation is that all the fields that can be related by $SU(N)$ duality relations, are:

- $N = 4$: $P^{*\,i\,IJ} = \frac{1}{2}\varepsilon^{IJKL} P_{i\,KL}$, and $\lambda_{iI} = \frac{1}{3!}\varepsilon_{IJKL}\lambda_i{}^{IJK}$.
- $N = 6$: $P^{*\,IJ} = \frac{1}{4!}\varepsilon^{IJK_1\cdots K_4} P_{K_1\cdots K_4}$, $\chi_{IJK} = \frac{1}{3!}\varepsilon_{IJKLMN}\lambda^{IJK}$, and $\chi^{I_1\cdots I_5} = \varepsilon^{I_1\cdots I_5 J}\lambda_J$.
- $N = 8$: $P^{*\,I_1\cdots I_4} = \frac{1}{4!}\varepsilon^{I_1\cdots I_4 J_1\cdots J_4} P_{J_1\cdots J_4}$, and $\chi_{I_1 I_2 I_3} = \frac{1}{5!}\varepsilon_{I_1 I_2 I_3 J_1\cdots J_5}\chi^{J_1\cdots J_5}$.
  These constraints must be taken into account in the action.

The scalars are encoded into the $2\bar{n}$-dimensional $(\bar{n} \equiv n + \frac{N(N-1)}{2})$ symplectic vectors

$$\mathcal{V}_{IJ} = \begin{pmatrix} f^{\Lambda}{}_{IJ} \\ h_{\Lambda\, IJ} \end{pmatrix}, \qquad \text{and} \qquad \mathcal{V}_{i} = \begin{pmatrix} f^{\Lambda}{}_{i} \\ h_{\Lambda\, i} \end{pmatrix}, \qquad \Lambda = 1, \cdots, \bar{n}\,,$$

normalized

$$\langle \mathcal{V}_{IJ} \mid \mathcal{V}^{*\,KL} \rangle = -2i\delta^{KL}{}_{IJ}\,, \qquad \langle \mathcal{V}_{i} \mid \mathcal{V}^{*\,j} \rangle = -i\delta_{i}{}^{j}\,.$$

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They can be combined into the $Usp(\bar{n}, \bar{n})$ matrix

$$U \equiv \tfrac{1}{\sqrt{2}} \begin{pmatrix} f + ih & f^{*} + ih^{*} \\ f - ih & f^{*} - ih^{*} \end{pmatrix}.$$

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The graviphotons $A^{IJ}{}_{\mu}$ do not appear directly, only through the "dressed" vectors

$$A^{\Lambda}{}_{\mu} \equiv \tfrac{1}{2} f^{\Lambda}{}_{IJ} A^{IJ}{}_{\mu} + f^{\Lambda}{}_{i} A^{i}{}_{\mu}\,.$$

The supersymmetry transformations of the fermioninc fields are

$$\begin{aligned}
\delta_{\epsilon}\psi_{I\mu} &= \mathfrak{D}_{\mu}\epsilon_{I} + T_{IJ}{}^{+}{}_{\mu\nu}\gamma^{\nu}\epsilon^{J}\,,\\
\delta_{\epsilon}\chi_{IJK} &= -\tfrac{3i}{2}\,\not{T}_{[IJ}{}^{+}\epsilon_{K]} + i\,\not{P}_{IJKL}\epsilon^{L}\,,\\
\delta_{\epsilon}\lambda_{iI} &= -\tfrac{i}{2}\,\not{T}_{i}{}^{+}\epsilon_{I} + i\,\not{P}_{iIJ}\epsilon^{J}\,,\\
\delta_{\epsilon}\chi_{IJKLM} &= -5i\,\not{P}_{[IJKL}\epsilon_{M]} + \tfrac{i}{2}\varepsilon_{IJKLMN}\,\not{T}^{-}\epsilon^{N} + \tfrac{i}{4}\varepsilon_{IJKLMNOP}\,\not{T}^{NO-}\epsilon^{P}\,,\\
\delta_{\epsilon}\lambda_{iIJK} &= -3i\,\not{P}_{i[IJ}\epsilon_{K]} + \tfrac{i}{2}\varepsilon_{IJKL}\,\not{T}_{i}{}^{-}\epsilon^{L} + \tfrac{i}{4}\varepsilon_{IJKLMN}\,\not{T}^{LM-}\epsilon_{N}\,,
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\mathfrak{D}_\mu \epsilon_I \equiv \nabla_\mu \epsilon_I - \epsilon_J \Omega_\mu{}^J{}_I \,,
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and $\Omega_\mu{}^J{}_I$ is the pullback of the connection of the scalar manifold $(\subset U(N))$.

The action for the bosonic fields is

$$S = \int d^4x \sqrt{|g|}\, \Big[R + 2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}F^{\Sigma}{}_{\mu\nu} - 2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}\star F^{\Sigma}{}_{\mu\nu}$$

$$+\tfrac{2}{4!}\alpha_1 P^{*\,IJKL}{}_{\mu}P_{IJKL}{}^{\mu} + \alpha_2 P^{*\,iIJ}{}_{\mu}P_{iIJ}{}^{\mu}\Big]\,,$$

where

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$$\begin{cases} \mathcal{E}^{IJKL} = \mathfrak{D}^{\mu}P^{*\,IJKL}{}_{\mu} + 6T^{[IJ|-}{}_{\mu\nu}T^{|KL]-\,\mu\nu} \\ \qquad\qquad +P^{*\,IJKL\,A}P^{*\,ij}{}_{A}T_{i}{}^{+}{}_{\mu\nu}T_{j}{}^{+\,\mu\nu}\,, \\ \mathcal{E}^{iIJ} = \mathfrak{D}^{\mu}P^{*\,iIJ}{}_{\mu} + T^{i\,-}{}_{\mu\nu}T^{IJ\,-\,\mu\nu} + \tfrac{1}{2}\varepsilon^{IJKL}T_{i}{}^{+}{}_{\mu\nu}T_{KL}{}^{+\,\mu\nu}\,. \end{cases}$$

For $N=5$ **:** $\mathcal{E}^{IJKL} = \mathfrak{D}^{\mu}P^{*\,IJKL}{}_{\mu} + 6T^{[IJ|-}{}_{\mu\nu}T^{|KL]-\,\mu\nu}$. etc.

9 – The all-N Killing Spinor Equations (KSEs)

For all values of N the independent KSEs take the form

$$
\begin{aligned}
\mathfrak{D}_\mu \epsilon_I + T_{IJ}{}^{+}{}_{\mu\nu}\gamma^\nu \epsilon^J &= 0\,,\\
\not{P}_{IJKL}\epsilon^L - \tfrac{3}{2}\not{T}_{[IJ}{}^{+}\epsilon_{K]} &= 0\,,\\
\not{P}_{i\,IJ}\epsilon^J - \tfrac{1}{2}\not{T}_i{}^{+}\epsilon_I &= 0\,,\\
\not{P}_{[IJKL}\epsilon_{M]} &= 0\,,\\
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Again, our goal is to find **all** the bosonic field configurations $\{e^a{}_\mu, A^\Lambda{}_\mu, P_{IJKL\,\mu}, P_{i\,IJ\,\mu}\}$ such that the above KSEs admit at least one solution ϵ^I.

10 – The all-N spinor-bilinears algebra

The independent bilinears that we can construct with one $U(N)$ vector of Weyl spinors ϵ_I are:

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2. $V_a \equiv V^I{}_{I\,a}$ is always non-spacelike: $V^2 = 2M^{IJ}M_{IJ} \equiv 2|M|^2 \geq 0$. We only consider the timelike case.
3. We can choose a tetrad $\{e^a{}_\mu\}$ such that $e^0{}_\mu \equiv \frac{1}{\sqrt{2}}|M|^{-1}V_\mu$. Then, **defining** $V^m{}_\mu \equiv |M|e^m{}_\mu$ we can decompose

$$V^I{}_{J\,\mu} = \tfrac{1}{2}\mathcal{J}^I{}_J V_\mu + \tfrac{1}{\sqrt{2}}(\sigma^m)^I{}_J V^m{}_\mu \,,$$

where $\mathcal{J}^I{}_J = 2M^{IK}M_{JK}|M|^{-2}$ is a rank 2 projector (Tod):

$$\mathcal{J}^2 = \mathcal{J}\,, \qquad \mathcal{J}^I{}_I = +2\,, \qquad \mathcal{J}^I{}_J\epsilon^J = \epsilon^I\,.$$

The main properties satisfied by the three σ^m matrices are:

$$\begin{aligned}
\sigma^m \sigma^n &= \delta^{mn} \mathcal{J} + i\varepsilon^{mnp}\sigma^p\,, \\
\mathcal{J}\sigma^m &= \sigma^m \mathcal{J} = \sigma^m\,, \\
(\sigma^m)^I{}_I &= 0\,, \\
\mathcal{J}^K{}_J \mathcal{J}^L{}_I &= \tfrac{1}{2}\mathcal{J}^K{}_I \mathcal{J}^L{}_J + \tfrac{1}{2}(\sigma^m)^K{}_I (\sigma^m)^L{}_J\,, \\
M_{K[I}(\sigma^m)^K{}_{J]} &= 0\,, \\
2|M|^{-2} M_{LI}(\sigma^m)^I{}_J M^{JK} &= (\sigma^m)^K{}_L\,,
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$$

$\{\mathcal{J}, \sigma^1, \sigma^2, \sigma^3\}$ is an x-dependent basis of a $\mathfrak{u}(2)$ subalgebra of $\mathfrak{u}(N)$ in the 2-dimensional eigenspace of $\mathcal{J}$ of eigenvalue $+1$ and provide a basis in the space of Hermitean matrices A satisfying $\mathcal{J} A \mathcal{J} = A$

11 – The all-N Killing Spinor Identities (KSIs)

If we assume that a given bosonic field configuration admits a Killing spinor ϵ_I, then we find that the (*off-shell*) "equations of motion" $\{\mathcal{E}^{\mu\nu}, \mathcal{E}^{\mu}, \mathcal{E}^{IJKL}, \mathcal{E}^{i\,IJ}\}$ satisfy the KSIs ($\tilde{\mathcal{J}}^{I}{}_{J} \equiv \delta^{I}{}_{J} - \mathcal{J}^{I}{}_{J}$):

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3. $$\left\{ \begin{array}{rcl} \mathcal{E}^{MNPQ}\mathcal{J}^{[I}{}_M\tilde{\mathcal{J}}^J{}_N\tilde{\mathcal{J}}^K{}_P\tilde{\mathcal{J}}^{L]}{}_Q & = & 0\,, \\ \mathcal{E}^{i\,MN}\mathcal{J}^{[I}{}_M\tilde{\mathcal{J}}^{J]}{}_N & = & 0\,, \end{array} \right.$$ ($\Rightarrow$ no attractor mechanism)

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$$N=4: \left\{ \begin{array}{rcl} \mathcal{E}^{IJKL} & = & -2\sqrt{2}\frac{M^{[IJ|}}{|M|}\langle\,\mathcal{E}^0 \mid \mathcal{V}^{*\,|KL]}\,\rangle\,, \\ \mathcal{E}_{iIJ} & = & -2\sqrt{2}\left\{\frac{M_{IJ}}{|M|}\langle\,\mathcal{E}^0 \mid \mathcal{V}_i\,\rangle + \frac{1}{2}\varepsilon_{IJKL}\frac{M^{KL}}{|M|}\langle\,\mathcal{E}^0 \mid \mathcal{V}^{*\,i}\,\rangle\right\}\,, \end{array}\right.$$

etc.

The only independent equations of motion that have to be imposed on **any** $d = 4$ supersymmetric configuration are

$$\mathcal{E}^0 = 0\,.$$

12 – The all-N supersymmetric solutions

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2. Choose three $N \times N$, Hermitean , traceless, x-dependent $(\sigma^m)^I{}_J$, satisfying the same properties as the Pauli matrices in the subspace preserved by $\mathcal{J}$.
   We also have to impose the constraint
   $$\mathcal{J} d\sigma^m \mathcal{J} = 0\,.$$

Once the $U(2)$ subgroup has been chosen, we can split the Vielbeins $P_{IJKL\,\mu}$ and $P_{i\,IJ\,\mu}$, into associated to the would-be vector multiplets in the $N = 2$ truncation

$$P_{IJKL}\,\mathcal{J}^{I}{}_{[M}\mathcal{J}^{J}{}_{N}\tilde{\mathcal{J}}^{K}{}_{P}\tilde{\mathcal{J}}^{L}{}_{Q]}\,, \quad \text{and} \quad P_{i\,IJ}\,\mathcal{J}^{I}{}_{[K}\mathcal{J}^{J}{}_{L]}\,,$$

which are driven by the *attractor mechanism* (*i.e.* they are determined by the electric and magnetic charges) and those associated to the hypermultiplets

$$P_{IJKL}\,\mathcal{J}^{I}{}_{[M}\tilde{\mathcal{J}}^{J}{}_{N}\tilde{\mathcal{J}}^{K}{}_{P}\tilde{\mathcal{J}}^{L}{}_{Q]}\,, \quad \text{and} \quad P_{i\,IJ}\,\mathcal{J}^{I}{}_{[K}\tilde{\mathcal{J}}^{J}{}_{L]}\,.$$

which are not.

In hyper-less solutions (*e.g.* black holes) the σ^{m}s matrices are not needed at all.

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where

$$|M|^{-2} \;=\; (M^{IJ}M_{IJ})^{-2} = \langle\, \mathcal{R} \mid \mathcal{I}\,\rangle ,$$

$$(d\omega)_{mn} \;=\; 2\epsilon_{mnp}\langle\, \mathcal{I} \mid \partial^p\mathcal{I}\,\rangle .$$

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Its spin connection $\varpi^{mn}$is related to Ω, by

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5. The vector field strengths are

$$F = -\tfrac{1}{2}d(\mathcal{R}\hat{V}) - \tfrac{1}{2}\star(\hat{V}\wedge d\mathcal{I})\,, \qquad \hat{V} = \sqrt{2}|M|^2(dt + \omega)\,.$$

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6. The scalars in the vector multiplets in the associated $N=2$ truncation

$$P_{IJKL}\,\mathcal{J}^I{}_{[M}\mathcal{J}^J{}_N\tilde{\mathcal{J}}^K{}_P\tilde{\mathcal{J}}^L{}_{Q]}\,, \quad \text{and} \quad P_{i\,IJ}\,\mathcal{J}^I{}_{[K}\mathcal{J}^J{}_{L]}\,,$$

can be found from $\mathcal{R}$ and $\mathcal{I}$, while those in the hypers must be found independently by solving

$$\begin{aligned} P_{IJKL\,m}\,\mathcal{J}^I{}_{[M}\tilde{\mathcal{J}}^J{}_N\tilde{\mathcal{J}}^K{}_P\tilde{\mathcal{J}}^L{}_{Q]}(\sigma^m)^Q{}_R &= 0\,, \\ P_{i\,IJ\,m}\,\mathcal{J}^I{}_{[K}\tilde{\mathcal{J}}^J{}_{L]}(\sigma^m)^L{}_M &= 0\,, \end{aligned}$$

which solve their equations of motion according to the *Killing Spinor Identities*.

13 – Final comments

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- ★ Much work remains to be done in order to make explicit the construction of the solutions. In particular one has to find general parametrizations of the matrices M^{IJ} and $\mathcal{J}^{I}{}_{J}$, solve the *stabilization equations*, impose the covariant constancy of $\mathcal{J}$ etc. (Meessen & O., work in progress).

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$$\mathcal{Z}[\phi(\rho), q] \equiv h^I(\phi) q_I\,.$$

Then, using $h^I h_I = 1$ and $dh^I h_I = h^I dh_I = 0$

$$df^{-1} = d(h^I h_I / f) = h^I d(h_I / f)\,,$$

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$$d\phi^x = h^{Ix} h_{Iy} d\phi^y = -\sqrt{3} h^{Ix} dh_I = -\sqrt{3} h^{Ix} d(f h_I / f) = -\sqrt{3} f h^{Ix} d(h_I / f)\,,$$

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The autonomous system of ordinary differential equations

$$\begin{cases} \dfrac{df^{-1}}{d\rho} &= \mathcal{Z}[\phi(\rho), q]\,, \\[2ex] \dfrac{d\phi^x}{d\rho} &= -f g^{xy}\partial_y \mathcal{Z}[\phi(\rho), q]\,. \end{cases}$$

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At the attractor point $\rho_{\rm attract}$ $\phi(\rho_{\rm attract}) = \phi_{\rm fix}$

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We define the central charges

$$\mathcal{Z}_{IJ}[\phi(\rho), q] \equiv \langle \mathcal{V}_{IJ} \mid q \rangle = p^{\Lambda} h_{\Lambda\, IJ} - q_{\Lambda} f^{\Lambda}{}_{IJ}\,,$$

$$\mathcal{Z}_{i}[\phi(\rho), q] \equiv \langle \mathcal{V}_{i} \mid q \rangle = p^{\Lambda} h_{\Lambda\, i} - q_{\Lambda} f^{\Lambda}{}_{i}\,.$$

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Then

$$\begin{aligned}
\mathfrak{D}\frac{M^{IJ}}{|M|^2} &= \mathfrak{D}\left(\frac{M^{KL}}{|M|^2}\tfrac{i}{2}\langle \mathcal{V}_{KL} \mid \mathcal{V}^{*\,IJ} \rangle\right) = \tfrac{i}{2}\mathfrak{D}\langle (\mathcal{R} + i\mathcal{I}) \mid \mathcal{V}^{*\,IJ} \rangle \\
&= \tfrac{i}{2}\langle d(\mathcal{R} + i\mathcal{I}) \mid \mathcal{V}^{*\,IJ} \rangle = \tfrac{i}{2}\langle d(\mathcal{R} - i\mathcal{I}) \mid \mathcal{V}^{*\,IJ} \rangle - \langle d\mathcal{I} \mid \mathcal{V}^{*\,IJ} \rangle \\
&= \tfrac{i}{2}\frac{M_{KL}}{|M|^2}\langle d\mathcal{V}^{*\,KL} \mid \mathcal{V}^{*\,IJ} \rangle - \langle q \mid \mathcal{V}^{*\,IJ} \rangle d\rho \\
&= \tfrac{1}{2}P^{*\,KLIJ}\frac{M_{KL}}{|M|^2} + \mathcal{Z}^{*\,IJ}[\phi(\rho), q]d\rho\,.
\end{aligned}$$

With the above identitiy we can compute

$$d|M|^{-2} = \frac{M_{IJ}}{|M|^2}\mathfrak{D}\frac{M^{IJ}}{|M|^2} + \frac{M^{IJ}}{|M|^2}\mathfrak{D}\frac{M_{IJ}}{|M|^2} = \frac{M_{IJ}\mathcal{Z}^{*\,IJ} + M^{IJ}\mathcal{Z}_{IJ}}{|M|^2}[\phi(\rho), q]d\rho\,,$$

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which leads to the flow equation ($N \geq 4$)

$$P^{*\,MN[IJ}\mathcal{J}^{K}{}_{M}\mathcal{J}^{L]}{}_{N} = -M^{[IJ}\mathcal{Z}^{*\,KL]}[\phi(\rho), q]d\rho\,.$$

The third flow equation ($N = 2, 3, 4, 6$) follows from

$$\begin{aligned}
\tfrac{1}{2}\frac{M^{IJ}}{|M|^2}P_{iIJ} &= -\tfrac{i}{2}\frac{M^{IJ}}{|M|^2}\langle\, d\mathcal{V}_{IJ} \mid \mathcal{V}_i \,\rangle = -\tfrac{i}{2}\langle\, d(\mathcal{R} + i\mathcal{I}) \mid \mathcal{V}_i \,\rangle \\
&= \langle\, d\mathcal{I} \mid \mathcal{V}_i \,\rangle - \tfrac{i}{2}\langle\, d(\mathcal{R} - i\mathcal{I}) \mid \mathcal{V}_i \,\rangle \\
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These flow equations lead to the generic *N attractor equations* (work in progress).