From tensor hierarchies to new supersymmetric solutions

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Plan of the Talk:

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⇒ By using the embedding tensor method to gauge arbitrary 4-dimensional FTs, we may be able to find all their $(p+1)$-form potentials, their democratic formulations and the extended objects (branes) that can couple to them.

What we are going to do in this seminar:

[a] So far, only maximal and half-maximal supergravities have been studied from this point of view de Wit, Samtleben & Trigiante, arXiv:hep-th/0412173, Samtleben & Weidner arXiv:hep-th/0506237, Schon & Weidner, arXiv:hep-th/0602024, de Wit, Samtleben & Trigiante, arXiv:0705.2101, Bergshoeff, Gomis, Nutma & Roest, arXiv:0711.2035, de Wit, Nicolai & Samtleben, arXiv:0801.1294. The only exception is de Vroome & de Wit arXiv:0707.2717, but the $U(2)$ R-symmetry group has not been properly taken into account.

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2. We will apply these results to $N = 1$ supergravity taking special care of the existence of $U(1)_R$ symmetry and a superpotential [a]. We will find all the $(p + 1)$-form potentials of $N = 1$ supergravity .

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4. The coupling of domain-wall sources to supergravity requires the introduction of a local coupling "constant" that gives rise to interesting effects.

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2 – The embedding tensor method and the tensor hierarchy

Consider an arbitrary ($N = 1$ supergravity -inspired) 4-dimensional ungauged FT with bosonic fields $\{Z^i, A^\Lambda\}$ (gravity plays no relevant role here)

$$S_{\rm u}[Z^i, A^\Lambda] = \int \{-2\mathcal{G}_{ij^*} dZ^i \wedge \star dZ^{*\,j^*} - 2\Im{\rm m} f_{\Lambda\Sigma} F^\Lambda \wedge \star F^\Sigma + 2\Re{\rm e} f_{\Lambda\Sigma} F^\Lambda \wedge F^\Sigma - \star V_{\rm u}(Z, Z^*)\}\,.$$

with $F^\Lambda \equiv dA^\Lambda$, the fundamental (electric) field strengths and $f_{\Lambda\Sigma} = f_{\Lambda\Sigma}(Z)$.

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In 4-d one can define magnetic (dual) 1-forms A_Λ: if the Maxwell equations are

$$dG_\Lambda = 0\,, \qquad \text{where} \qquad G_\Lambda{}^+ \equiv f_{\Lambda\Sigma} F^{\Sigma\,+}\,,$$

then we can replace them by the duality relations (+ Bianchi identity)

$$G_\Lambda = F_\Lambda\,, \qquad \text{where} \qquad F_\Lambda \equiv dA_\Lambda\,.$$

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Furthermore, it (its equations of motion) may have non-perturbative global symmetries, including electric -magnetic duality rotations:

$$\begin{aligned}
\delta_{\alpha} Z^{i} &= \alpha^{A} k_{A}{}^{i}(Z)\,,\\
\delta_{\alpha} f_{\Lambda\Sigma} &= \alpha^{A}\{-T_{A\,\Lambda\Sigma} + 2T_{A\,(\Lambda}{}^{\Omega} f_{\Sigma)\Omega} - T_{A}{}^{\Omega\Gamma} f_{\Omega\Lambda} f_{\Gamma\Sigma}\}\,,\\
\delta_{\alpha} \begin{pmatrix} A^{\Lambda} \\ A_{\Lambda} \end{pmatrix} &= \alpha^{A} \begin{pmatrix} T_{A\,\Sigma}{}^{\Lambda} & T_{A}{}^{\Sigma\Lambda} \\ T_{A\,\Sigma\Lambda} & T_{A}{}^{\Sigma}{}_{\Lambda} \end{pmatrix} \begin{pmatrix} A^{\Sigma} \\ A_{\Sigma} \end{pmatrix}.
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\end{aligned}$$

The T_A matrices either belong to $\mathfrak{sp}(2n_V,\mathbb{R})$ or vanish (Gaillard & Zumino). We introduce the symplectic notation

$$A^M \equiv \begin{pmatrix} A^\Sigma \\ A_\Sigma \end{pmatrix} \qquad (T_{A\,M}{}^N) \equiv \begin{pmatrix} T_{A\,\Sigma}{}^\Lambda & T_A{}^{\Sigma\Lambda} \\ T_{A\,\Sigma\Lambda} & T_A{}^\Sigma{}_\Lambda \end{pmatrix}, \qquad [T_A, T_B] = -f_{AB}{}^C T_C\,.$$

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Each embedding tensor $\vartheta_M{}^A$ defines a possible set of identifications:

$$\alpha^A(x) \equiv \Lambda^\Sigma \vartheta_\Sigma{}^A + \Lambda_\Sigma \vartheta^{\Sigma\, A}\,, \qquad\qquad A^A \equiv A^\Sigma \vartheta_\Sigma{}^A + A_\Sigma \vartheta^{\Sigma\, A}\,.$$

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$\vartheta_M{}^A$ is subject to several constraints. First of all, the electric and magnetic charges must be *mutually local* (de Wit, Samtleben & Trigiante, arXiv:`hep-th/0507289`) satisfying the second quadratic constraint:

$$Q^{AB} \equiv \tfrac{1}{4}\vartheta^{MA}\vartheta_M{}^B = 0\,.$$

If we try to construct the covariant derivatives $\mathfrak{D}$

$$\mathfrak{D}Z^i \equiv dZ^i + A^M \vartheta_M{}^A k_A{}^i \,,$$

they will only transform covariantly under

$$\delta_\Lambda Z^i = \Lambda^M \vartheta_M{}^A k_A{}^i(Z) \,,$$

$$\delta_\Lambda A^M = -\mathfrak{D}\Lambda^M \equiv -(d\Lambda^M + \vartheta_N{}^A T_{A\,P}{}^M A^N \Lambda^P) \,,$$

if $\vartheta_M{}^A$ is an invariant tensor

$$\delta_\Lambda \vartheta_M{}^A = -\Lambda^N Q_{MN}{}^A = 0 \,, \qquad Q_{MN}{}^A \equiv \vartheta_M{}^B T_{B\,N}{}^P \vartheta_P{}^A - \vartheta_M{}^B \vartheta_N{}^C f_{BC}{}^A \,.$$

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Finally, we must impose another (*linear* or *representation constraint*) on top of the two quadratic ones $Q_{MN}{}^A = Q^{AB} = 0$:

$$L_{MNP} \equiv \vartheta_{(M}{}^A T_{A\,NP)} = 0 \,.$$

We can't construct gauge -covariant 2-form field strengths F^M without it!

The simultaneous use of electric and magnetic 1-forms as gauge fields has an importance consequence: we have to modify the *naive* 2-form field strengths

$$F^{M}_{\text{naive}} = dA^{M} + \tfrac{1}{2}\vartheta_{N}{}^{A}T_{AP}{}^{M}A^{N}\wedge A^{P}\,,$$

by adding a Stückelberg coupling to a 2-form B_A:

$$F^{M} = dA^{M} + \tfrac{1}{2}\vartheta_{N}{}^{A}T_{AP}{}^{M}A^{N}\wedge A^{P} + Z^{MA}B_{A}\,, \qquad Z^{MA} \equiv -\tfrac{1}{2}\vartheta^{MA}\,.$$

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In this way, for any 4-d Field Theory, we always obtain a tower of $(p+1)$-forms $A^M, B_A, C_C{}^M, D_{AB}, D^{NPQ}, D_E{}^{NP}$ related by gauge transformations:

The (generic, bosonic , 4-dimensional) tensor hierarchy .

But, what does it mean?

What is the meaning of the additional fields?

3 – The meaning of the $d = 4$ tensor hierarchy

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However, gauging must not introduce new continuous degrees of freedom in a FT: for $p \leq d - 3$ they must be related by duality relations to the fundamental ones.

These duality relations together with the 1st order Bianchi identities

$$\mathfrak{D}\mathfrak{D}Z^{i} = F^{M}\vartheta_{M}{}^{A}k_{A}{}^{i} \ .$$

$$\mathfrak{D}F^{M} = Z^{MA}H_{A} \ .$$

$$\mathfrak{D}H_{A} = T_{A\,MN}F^{M} \wedge F^{N} + Y_{AM}{}^{C}G_{C}{}^{M} \ .$$

must give the 2nd order equations of motion.

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☞ These two duality relations together with the Bianchi identity $\mathfrak{D}F^M = Z^{MA} H_A$ give a set of electric -magnetic duality -covariant Maxwell equations:

$$\mathfrak{D}F^\Lambda = -\tfrac{1}{4}\vartheta_\Lambda{}^A \star j_A \ , \qquad \mathfrak{D}G_\Lambda = \tfrac{1}{4}\vartheta^{\Lambda\, A} \star j_A \ .$$

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- The 3-forms $C_C{}^M$ must be "*dual to constants*", i.e. to the deformation parameters. Their indices are indeed conjugate to those of the embedding tensor $\vartheta_M{}^C$. This duality is expressed through the formula

$$G_C{}^M = \tfrac{1}{2} \star \frac{\partial V}{\partial \vartheta_M{}^C} \ .$$

☞ Using the three duality relations in the Bianchi identity of H_A we get

$$\mathfrak{D} \star j_A = 4T_{A\,MN} G^M \wedge G^N + \star Y_A{}^{MC} \frac{\partial V}{\partial \vartheta_M{}^C} \,.$$

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This equation is similar to the consistency condition (gauge or Noether identity) that Noether currents must satisfy off-shell in FTs with gauge invariance:

$$\mathfrak{D} \star j_A = -2(k_A{}^i \mathcal{E}_i + \text{c.c.}) + 4T_{A\,MN} G^M \wedge G^N + \star Y_A{}^{MC} \frac{\partial V}{\partial \vartheta_M{}^C} \, ,$$

where $\mathcal{E}_i$ is the e.o.m. of Z^i. Both equations, together, imply

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which is equivalent to the scalar e.o.m. for symmetric σ-models.

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☞ Finally, the indices of the three 4-forms $D_{AB}, D^{NPQ}, D_E{}^{NP}$ are conjugate to those of the constraints $Q^{AB}, Q_{NPQ}, Q_{NP}{}^E$. They are Lagrange multipliers enforcing them.

This interpretation is confirmed by the existence of a gauge -invariant (*democratic*) action for **all** these fields, (including the embedding tensor $\vartheta_M{}^A(x)$!):

This gauge -invariant action is given by

$$S[g_{\mu\nu}, Z^i, A^M, B_A, C_A{}^M, D_E{}^{NP}, D_{AB}, D^{MNP}, \vartheta_M{}^A] \quad =$$

$$\int \{-2\mathcal{G}_{ij^*}\mathfrak{D}Z^i \wedge \star\mathfrak{D}Z^{*\,j^*} + 2F^\Sigma \wedge G_\Sigma - \star V$$

$$-4Z^{\Sigma A} B_A \wedge \left(F_\Sigma - \tfrac{1}{2} Z_\Sigma{}^B B_B\right) - \tfrac{4}{3} X_{[MN]\Sigma} A^M \wedge A^N \wedge \left(F^\Sigma - Z^{\Sigma B} B_B\right)$$

$$-\tfrac{2}{3} X_{[MN]}{}^\Sigma A^M \wedge A^N \wedge \left(dA_\Sigma - \tfrac{1}{4} X_{[PQ]\Sigma} A^P \wedge A^Q\right)$$

$$-2\mathfrak{D}\vartheta_M{}^A \wedge (C_A{}^M + A^M \wedge B_A)$$

$$+2Q_{NP}{}^E(D_E{}^{NP} - \tfrac{1}{2} A^N \wedge A^P \wedge B_E) + 2Q^{AB} D_{AB} + 2L_{MNP} D^{MNP}\} \,.$$

4 – Application: general gaugings of $N = 1, d = 4$ supergravity

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- In $N = 1 \quad N = 2$ supergravity one can write $G = G_{\text{bos}} \times H_{\text{aut}}$, i.e. R-symmetry only acts on the fermions , which have been ignored in the construction of the *universal* tensor hierarchy .
- $N = 1$ supergravity can be deformed with an arbitrary holomorphic superpotential $\mathcal{W}(Z)$ which appears through the covariantly holomorphic section of Kähler weight $(1, -1)$ $\mathcal{L}(Z, Z^*)$:

$$\mathcal{L}(Z, Z^*) = \mathcal{W}(Z) e^{\mathcal{K}/2} , \qquad \mathcal{D}_{i^*} \mathcal{L} = 0 ,$$

coupling to the fermions in various ways and gives rise to the scalar potential

$$V_{\text{u}}(Z, Z^*) = -24|\mathcal{L}|^2 + 8\mathcal{G}^{ij^*} \mathcal{D}_i \mathcal{L} \mathcal{D}_{j^*} \mathcal{L}^* .$$

☞ When $\mathcal{L}(Z, Z^*) \neq 0$, we must transform it under $U(1)_R$ to leave the theory invariant. However, the scalars Z^i are inert under $U(1)_R$ and we can only accept the transformation of $\mathcal{L}(Z, Z^*)$ if we can associate it to **another** kind of symmetry which acts on the scalars.

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We want to find all these new potentials and their supersymmetry transformations to find possible new supersymmetric extended objects (branes) in $N = 1$ supergravity .

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The bosonic action is the one we have considered before

$$S_{\rm u}[g_{\mu\nu}, Z^i, A^{\Lambda}] = \int \{\star R - 2\mathcal{G}_{ij^*} dZ^i \wedge \star dZ^{*\,j^*} - 2\Im{\rm m} f_{\Lambda\Sigma} F^{\Lambda} \wedge \star F^{\Sigma}$$

$$+2\Re{\rm e} f_{\Lambda\Sigma} F^{\Lambda} \wedge F^{\Sigma} - \star V_{\rm u}(Z, Z^*)\}\,.$$

with $\mathcal{G}_{ij^*} = \partial_i \partial_{j^*} \mathcal{K}$ and $f_{\Lambda\Sigma} = f_{\Lambda\Sigma}(Z)$.

6 – Gauging $N = 1, d = 4$ supergravity

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We expect changes in the standard $d = 4$ tensor hierarchy (at least new 3-form C and 4-forms D^M) which have to be confirmed by checking supersymmetry .

7 – The $N = 1, d = 4$ supersymmetric tensor hierarchy

As a first step to include the tensor hierarchy fields into $N = 1$ supergravity we have constructed supersymmetry transformation rules such that the local supersymmetry algebra, to leading order in fermions , closes on the new fields up to duality relations (Hartong, Hübscher, O. arXiv:0903.0509) .

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- ☞ There is a new 4-form D^M associated to the new constraint Q_M.

The 3-forms C^1, C^2

We find a complex 3-form $\mathcal{C}_{\mu\nu\rho} = C^1{}_{\mu\nu\rho} + iC^2{}_{\mu\nu\rho}$ with

$$\delta_\epsilon \mathcal{C}_{\mu\nu\rho} = 12i\mathcal{L}\,\bar{\epsilon}^*\gamma_{[\mu\nu}\psi^*{}_{\rho]} + 2\mathcal{D}_i\mathcal{L}\bar{\epsilon}^*\gamma_{\mu\nu\rho}\chi^i + \text{c.c.}$$

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Replacing everywhere $\mathcal{L} \longrightarrow (\mathbf{g^1} + i\mathbf{g^2})\mathcal{L}$ where $\mathbf{g^1}$ and $\mathbf{g^2}$ are two *coupling constants*, the local supersymmetry algebra closes upon the duality relation

$$d\mathcal{C} = (\mathbf{g^1} + i\mathbf{g^2}) \star (-24|\mathcal{L}|^2 + 8\mathcal{G}^{ij^*}\mathcal{D}_i\mathcal{L}\mathcal{D}_{j^*}\mathcal{L}^*)\,, \quad \text{or} \quad dC^i = \tfrac{1}{2} \star \frac{\partial V}{\partial \mathbf{g^i}}\,, \quad i = 1, 2\,.$$

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The 4-forms D^M

In the ungauged $\vartheta_M{}^A = 0$ case when there are no symmetries acting on the 1-forms i.e. $T_{A\,M}{}^N = 0$ (for simplicity) the supersymmetry transformations are

$$\delta_\epsilon D^M = -\tfrac{i}{2}\star\mathcal{L}^*\bar{\epsilon}\lambda^M + \text{c.c.} + C\wedge\delta_\epsilon A^M\,.$$

8 – The supersymmetric objects of $N = 1$ supergravity

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We are going to focus on the domain walls associated to the 3-form C^1 $(\mathbf{g^2} = 0)$. We consider the ungauged theory with only chiral supermultiplets and superpotential

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The first-order *flow equations* imply the second-order supergravity e.o.m..

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$$S_{\text{DW}} = -\int d^3\xi\left\{|\mathcal{L}|\sqrt{|g_{(3)}|} \pm \tfrac{1}{4!}\epsilon^{mnp}C_{mnp}\right\}\,,$$

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In the static gauge $\partial X^\mu/\partial\xi^m = \delta^\mu{}_m$ it can be seen that this action is invariant to lowest order in fermions under the supersymmetry transformations of $g_{\mu\nu}, Z^i, C'_{\mu\nu\rho}$ if the spinors satisfy the BPS domain-wall projection $(e^{-i\alpha/2}\epsilon) \pm i\gamma^{012}(e^{-i\alpha/2}\epsilon)^* = 0$.

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and the brane source action

$$S_{\text{brane}} = -\int d^4x\,\mathbf{f}(y)\left\{|\mathcal{L}|\sqrt{|g_{(3)}|} \pm \tfrac{1}{4!}\epsilon^{mnp}C_{\underline{mnp}}\right\}\,,$$

where $\mathbf{f}(y)$ is a distribution function of the domain walls' common transverse direction $x^3 \equiv y$: $\mathbf{f}(y) = \delta^{(1)}(y-y_0)$ for a single domain wall placed at $y=y_0$ etc.

The equations of motion that follow from $S \equiv S_{\text{bulk}} + S_{\text{brane}}$ are

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\begin{aligned}
\mathcal{E}_{\mathbf{g}}^{\mu\nu} &= -\frac{\kappa^2}{2}\mathbf{f}(y)|\mathcal{L}|\frac{\sqrt{|g_{(3)}|}}{\sqrt{|g|}}g_{(3)}^{mn}\delta_m{}^{\mu}\delta_n{}^{\nu}\,, \\
\mathcal{G}^{ij^*}\mathcal{E}_{\mathbf{g}\,i^*} &= -\frac{\kappa^2}{8}\mathbf{f}(y)\frac{\sqrt{|g_{(3)}|}}{\sqrt{|g|}}e^{i\alpha}\mathcal{N}^i\,, \\
\epsilon^{\mu\nu\rho\sigma}\partial_\sigma\mathbf{g}(x) &= \pm\frac{\kappa^2}{8}\mathbf{f}(y)\epsilon^{mnp}\delta_m{}^{\mu}\delta_n{}^{\nu}\delta_p{}^{\rho}\,, \\
\epsilon^{\mu\nu\rho\sigma}\partial_\mu C_{\nu\rho\sigma} &= 6\mathbf{g}(x)V(Z,Z^*)\,,
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where $\mathcal{E}_{\mathbf{g}}^{\mu\nu}$ and $\mathcal{E}_{\mathbf{g}\,i^*}$ are the Einstein and scalar equations of motion with $\mathbf{g}(x) \neq 0$.

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The third equation is that of the 3-form and is solved if $\mathbf{g}$ is a function of y satisfying

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The fourth equation ($\mathbf{g}(x)$') states that C is the dual of the scalar potential.

The Einstein and scalar equations of motion with sources are identically satisfied if $H(y)$ and the scalars $Z^i(y)$ satisfy the *sourceful flow equations*

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\partial_{\underline{y}} Z^i &= \pm \mathbf{g}(y) e^{i\alpha} \mathcal{N}^i H^{1/2}\,, \\
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A fully supersymmetric "democratic" formulation of $N=1$ $d=4$ supergravity including all higher-rank forms and local coupling constants $\vartheta_M{}^A(x), \mathbf{g^1}(x), \mathbf{g^2}(x)$ is necessary to accomodate these modifications.

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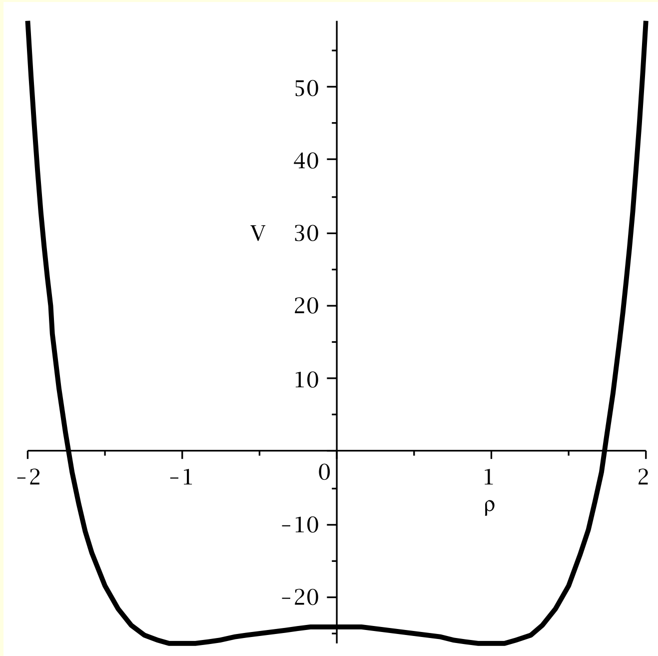
Observe that the space-dependent coupling constant $\mathbf{g}(x)$, sourced by domain walls , may modify the effective scalar potential dramatically.

12 – A simple example

Let us consider the model (1 chiral multiplet) defined by

$$\mathcal{K} = |Z|^2\,, \qquad \mathcal{W} = 1\,, \qquad (\mathcal{L} = e^{|Z|^2/2}\,,\ \mathcal{N}^Z = 2Z^* e^{|Z|^2/2})\,.$$

These choices lead to the Mexican-hat-type potential $V = -8(3 - \rho^2)e^{\rho^2/2}$ $(\rho \equiv |Z|)$



The *sourceful flow equations* take the form ($\mathrm{Arg}\, Z = \text{const}$)

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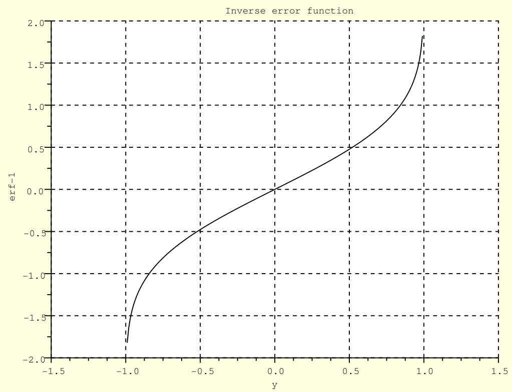
II-b Solutions with $\mathbf{g} \neq 0$ and $\partial_{\underline{y}} Z \neq 0$:

$$H = c/\rho^2\,,$$

$$\rho = \sqrt{2}\,\mathrm{erf}^{-1}\left[\mathbf{G}(y)\right]\,, \qquad \mathbf{G}(y) \equiv \pm\sqrt{\tfrac{8c}{\pi}}\int \mathbf{g}(y)dy + d\,.$$

erf^{-1} is the inverse of the *normalized error function* $\text{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$



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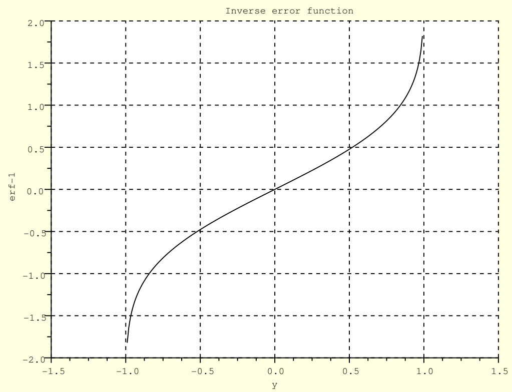

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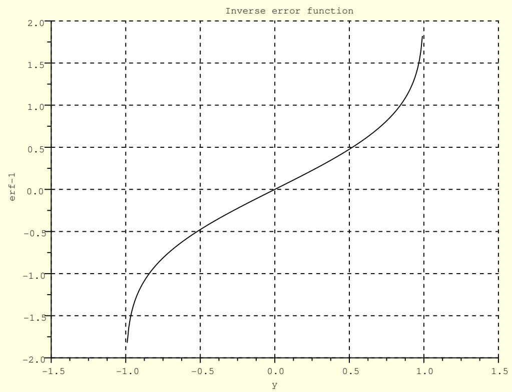

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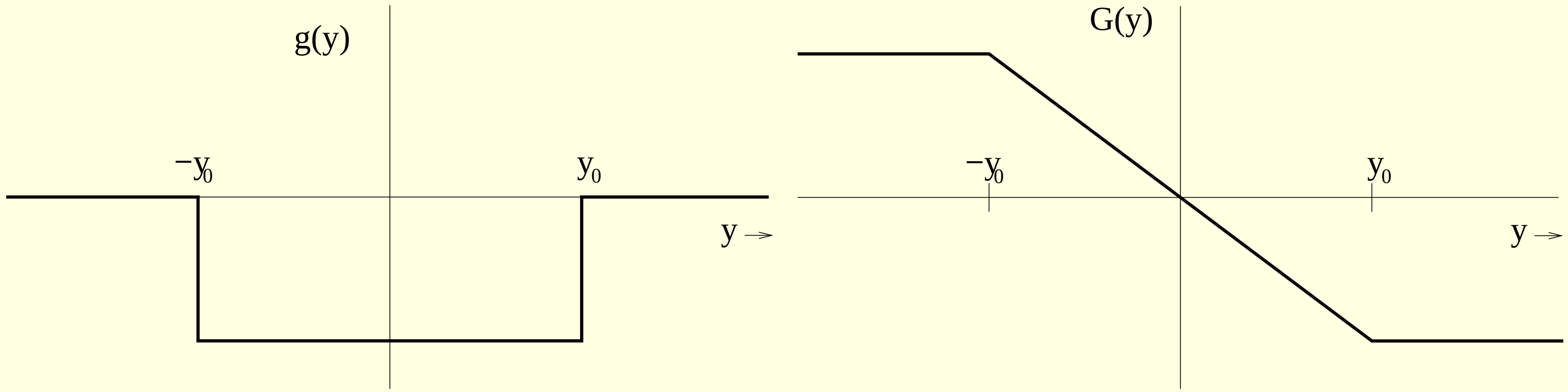
The simplest possibility is a single, infinitely thin domain-wall source of tension $q > 0$ placed at $y = y_0$:

$$\mathbf{f}(y) = q\delta(y - y_0)\,, \qquad \mathbf{g}(y) = \pm\frac{\kappa^2 q}{16}[\theta(y-y_0) - \theta(y_0 - y)]\,, \qquad \mathbf{G}(y) = \frac{\sqrt{c}\kappa^2 q}{\sqrt{32\pi}}|y - y_0| + d\,.$$

$\mathbf{G}(y)$ is always unbounded we have to cut the space by hand.

A possible solution: we introduce two parallel domain walls with opposite tension (a Randall-Sundrum-like construction) and charge at a different point ($y = -y_0$ with $y_0 > 0$ for simplicity) so

$$\begin{aligned}
\mathbf{f}(y) &= q\delta(y - y_0) - q\delta(y + y_0)\,,\\
\mathbf{g}(y) &= \pm\frac{\kappa^2 q}{16}[\theta(y - y_0) - \theta(y_0 - y) - \theta(y + y_0) + \theta(-y_0 - y)]\,,\\
\mathbf{G}(y) &= \sqrt{\frac{c}{32\pi}}\kappa^2 q\,(|y - y_0| - |y + y_0|) + d\,.
\end{aligned}$$



Choosing $d = \sqrt{\frac{c}{8\pi}}\kappa^2 q y_0$ we can set $\mathbf{G}(+\infty) = \mathbf{G}(+y_0) = 0$ and $\rho(y) = \rho(+y_0) = 0$ for $y > y_0$.

In the interior of the $\mathbf{g}(y) \neq 0$ region ρ approaches zero as $\rho \sim \frac{1}{4}\sqrt{c}\kappa^2 q(y_0 - y)$ so the metric approaches AdS_4

$$H \sim \frac{R^2}{(y_0 - y)^2}\,, \qquad R = \frac{4}{\kappa^2 q}\,.$$

The value $\mathbf{G}(-y_0) = \sqrt{\frac{c}{2\pi}}\kappa^2 q y_0 = \mathbf{G}(-\infty)$, can be tuned by varying distance between the domain-wall sources (y_0). It has to be smaller or equal than 1.

If $\mathbf{G}(-y_0) < 1$ then $\rho(-y_0)$ is finite and ρ approaches $y = -y_0$ from the interior of the $\mathbf{g}(y) \neq 0$ region as

$$\rho \sim -\sqrt{\frac{c}{2\pi}}\frac{\kappa^2 q}{\text{erf}'[\rho(-\infty)/\sqrt{2}]}(y + y_0)\,,$$

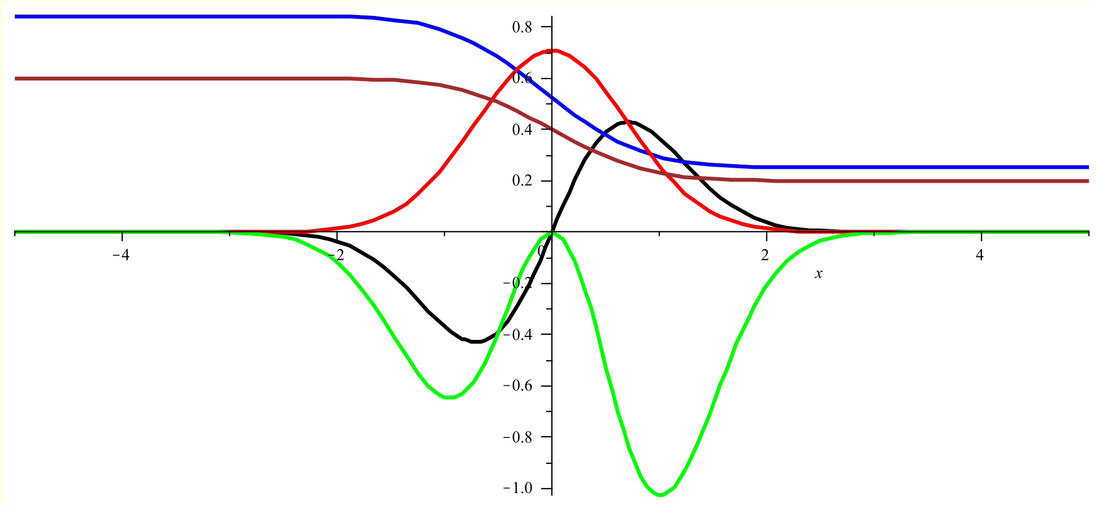
so the metric approaches another AdS_4 region.

This solution we have obtained smoothly interpolates between two AdS_4 regions one of which (the $\rho = 0$ one) corresponds to a supersymmetric vacuum of the theory.

The two infinitely-thin domain-wall system can be understood as an approximation to a configuration with domain-wall sources of finite thickness such as this:

$$\mathbf{f}(y) = qye^{-y^2}\,, \qquad \mathbf{g}(y) = \mp\frac{\kappa^2 q}{16}e^{-y^2}\,, \qquad \mathbf{G}(y) = -\frac{\kappa^2 q\sqrt{c}}{8}\mathrm{erf}(y) + d\,.$$

in which $\mathbf{g}(y)$ only vanishes asymptotically.



The profiles of some of the functions ocurring in this solution: the **black line**: the source, $\mathbf{f}(y)$, **red line**: the coupling constant $\mathbf{g}(y)$, **brown line** $\mathbf{G}(y)$, **blue line**: the scalar $\rho(y)$, **green line**: the effective potential as seen by the solution, *i.e.* $\mathbf{g^2}(\mathbf{y})V$.

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- ★ We have constructed the effective actions for the domain walls associated to C^1 and we have used them as sources for the bulk supergravity action. We have shown how the supersymmetry rules may be modified in a fully *democratic* formulation of $N = 1$ supergravity .
- ★ We have seen that in some cases domain-wall sources have to be introduced to construct sensible domain-wall solutions. These sources introduce a spacetime-dependent coupling constant $\mathbf{g}(x)$ that can have dramatic effects on the form of the solutions.

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$$\delta_\eta \chi^i = i \not{\mathfrak{D}} Z^i \eta^* + 2\mathcal{G}^{ij^*}\mathcal{D}_{j^*}\mathcal{L}^*\eta\,, \qquad \mathfrak{D} Z^i = dZ^i + A^M \vartheta_M{}^A k_A{}^i\,.$$

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We find the expected result

$$\begin{aligned}
[\delta_\eta\,,\,\delta_\epsilon] Z^i &= \delta_{\text{g.c.t.}} Z^i + \delta_h Z^i\,,\\
\delta_{\text{g.c.t.}} Z^i &= \pounds_\xi Z^i = +\xi^\mu \partial_\mu Z^i\,,\\
\delta_h Z^i &= \Lambda^M \vartheta_M{}^A k_A{}^i\,,\\
\xi^\mu &\equiv \tfrac{i}{4}(\bar{\epsilon}\gamma^\mu \eta^* - \bar{\eta}\gamma^\mu \epsilon^*)\,,\\
\Lambda^M &\equiv \xi^\mu A^M{}_\mu\,.
\end{aligned}$$

The 1-forms A^M

We introduce supersymmetric partners λ_Σ for the magnetic 1-forms A_Σ and make the symplectic -covariant *Ansatz*

$$\begin{aligned} \delta_\epsilon A^M{}_\mu &= -\tfrac{i}{8}\bar{\epsilon}^*\gamma_\mu \lambda^M + \text{c.c.}\,, \\ \delta_\epsilon \lambda^M &= \tfrac{1}{2}\left[\not{F}^{M+} + i\mathcal{D}^M\right]\epsilon\,, \end{aligned}$$

where we have defined the symplectic vector

$$\mathcal{D}^M \equiv \begin{pmatrix} \mathcal{D}^\Lambda \\ \mathcal{D}_\Lambda \end{pmatrix} \equiv \begin{pmatrix} \mathcal{D}_\Lambda \\ f_{\Lambda\Sigma}\mathcal{D}^\Sigma \end{pmatrix}\,, \qquad \mathcal{D}^\Lambda = -\Im\text{m} f^{\Lambda\Sigma}\,(\vartheta_\Sigma{}^A + f^*_{\Sigma\Omega}\vartheta^{\Omega\,A})\mathcal{P}_A\,.$$

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but **we do not need them** to show that

$$[\delta_\eta\,,\,\delta_\epsilon]\,A^M = \delta_{\text{g.c.t.}}A^M + \delta_h A^M\,,$$

where

$$\Lambda_A \equiv -T_{A\,MN}A^N\Lambda^M + b_A - \mathcal{P}_A\xi\,, \qquad b_{A\,\mu} \equiv B_{A\,\mu\nu}\xi^\nu\,.$$

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We introduce the supersymmetric partners of the 2-forms $B_{A\,\mu\nu}$ ζ_A, φ_A (linear supermultiplets)

$$\begin{aligned}
\delta_\epsilon \zeta_A &= -i\left[\tfrac{1}{12}\not{H}'_A + \not{\mathfrak{D}}\varphi_A\right]\epsilon^* - 4\delta_A{}^{\mathbf{a}}\varphi_{\mathbf{a}}\mathcal{L}^*\epsilon\,,\\
\delta_\epsilon B_{A\,\mu\nu} &= \tfrac{1}{4}\left[\bar{\epsilon}\gamma_{\mu\nu}\zeta_A + \text{c.c.}\right] - i\left[\varphi_A\bar{\epsilon}^*\gamma_{[\mu}\psi_{\nu]} - \text{c.c.}\right] + 2T_{A\,MN}A^M{}_{[\mu}\delta_\epsilon A^N{}_{\nu]}\,,\\
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but, again, **we do not need them** to show that

$$[\delta_\eta\,,\,\delta_\epsilon]\,B_A = \delta_{\text{g.c.t.}}B_A + \delta'_h B_A\,,$$

which **proves** the existence of an extra Stückelberg shift in B_A.

The 3-forms $C_A{}^M$

In this case we won't introduce supersymmetric partners. We make the *Ansatz*

$$\delta_\epsilon C_A{}^M{}_{\mu\nu\rho} = -\tfrac{i}{8}\left[\mathcal{P}_A \bar{\epsilon}^* \gamma_{\mu\nu\rho}\lambda^M - \text{c.c.}\right] - 3B_{A\,[\mu\nu|}\delta_\epsilon A^M{}_{|\rho]} - 2T_{A\,PQ}A^M{}_{[\mu}A^P{}_{\nu|}\delta_\epsilon A^Q{}_{|\rho]}\,.$$

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This corresponds to th manifestly symplectic -invariant scalar potential

$$V_{\mathrm{e-mg}} = V_{\mathrm{u}} - \tfrac{1}{2}\Re\mathrm{e}\,\mathcal{D}^M\vartheta_M{}^A\mathcal{P}_A = V_{\mathrm{u}} + \tfrac{1}{2}\mathcal{M}^{MN}\vartheta_M{}^A\vartheta_N{}^B\mathcal{P}_A\mathcal{P}_B\,,$$

where

$$(\mathcal{M}^{MN}) \equiv \begin{pmatrix} I^{\Lambda\Sigma} & I^{\Lambda\Omega}R_{\Omega\Sigma} \\ R_{\Lambda\Omega}I^{\Omega\Sigma} & I_{\Lambda\Sigma} + R_{\Lambda\Omega}I^{\Omega\Gamma}R_{\Gamma\Sigma}\end{pmatrix}, \qquad \begin{aligned} f_{\Lambda\Sigma} &\equiv R_{\Lambda\Sigma} + iI_{\Lambda\Sigma}\,, \\ I^{\Lambda\Omega}I_{\Omega\Sigma} &\equiv \delta^\Lambda{}_\Sigma\,. \end{aligned}$$