The tensor hierarchy and supersymmetric domain walls of N=1,d=4 supergravity

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Talk given on the 29th of October 2009 at the *Departamento de Física Teórica e Historia de la Ciencia, UPV/EHU, Leioa, Bilbao*

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Work done in collaboration with *E. Bergshoeff, O. Hohm* (U. Groningen) *J. Hartong* (U. Bern) *M. Hübscher* and *P. Meessen* (IFT UAM/CSIC, Madrid)

Plan of the Talk:

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3. The *embedding tensor method* (Cordaro, Fré, Gualtieri,Termonia & Trigiante, arXiv:`hep-th/9804056`.) can be used to construct systematically the most general gauged supergravities . This construction requires the introduction of additional $(p+1)$-form potentials.

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We are going to use the embedding tensor method to find all the $(p+1)$-form potentials and the corresponding democratic formulations of any 4-dimensional FT with gauge symmetry and we are going to apply the general results to the particular case of $N=1$ supergravity .

What we are going to do in this seminar:

[a] So far, only maximal and half-maximal supergravities have been studied from this point of view de Wit, Samtleben & Trigiante, arXiv:hep-th/0412173, Samtleben & Weidner arXiv:hep-th/0506237, Schon & Weidner, arXiv:hep-th/0602024, de Wit, Samtleben & Trigiante, arXiv:0705.2101, Bergshoeff, Gomis, Nutma & Roest, arXiv:0711.2035, de Wit, Nicolai & Samtleben, arXiv:0801.1294. The only exception is de Vroome & de Wit arXiv:0707.2717, but the $U(2)$ R-symmetry group has not been properly taken into account.

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5. We will use some of the new $(p + 1)$-form potentials to construct supersymmetric p-brane effective actions and solutions with sources of $N = 1$ supergravity .

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2 – The embedding tensor method: electric gaugings

Consider a general ($N = 1$ supergravity -inspired) 4-dimensional ungauged FT with bosonic fields $\{Z^i, A^\Lambda\}$ (the metric plays no relevant role here)

$$S_{\rm u}[Z^i, A^\Lambda] = \int \{-2\mathcal{G}_{ij^*}dZ^i\wedge\star dZ^{*\,j^*} - 2\Im{\rm m} f_{\Lambda\Sigma}F^\Lambda\wedge\star F^\Sigma + 2\Re{\rm e} f_{\Lambda\Sigma}F^\Lambda\wedge F^\Sigma - \star V_{\rm u}(Z, Z^*)\}\,.$$

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Let us assume this action is invariant under the global transformations

$$\begin{aligned}
\delta_\alpha Z^i &= \alpha^A k_A{}^i(Z)\,,\\
\delta_\alpha f_{\Lambda\Sigma} &\equiv -\alpha^A \pounds_A f_{\Lambda\Sigma} = \alpha^A[T_{A\,\Lambda\Sigma} - 2T_{A\,(\Lambda}{}^\Omega f_{\Sigma)\Omega}]\,,\\
\delta_\alpha A^\Lambda &= \alpha^A T_{A\,\Sigma}{}^\Lambda A^\Sigma\,.
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Each embedding tensor ${\vartheta_\Lambda}^A$ defines a possible identification:

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covariant under

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This only works if ${\vartheta_{\Lambda}}^{A}$ is an invariant tensor

$$\delta_{\Lambda}{\vartheta_{\Sigma}}^{A} = -\Lambda^{\Omega}{Q_{\Omega\Sigma}}^{A} = 0\,, \qquad {Q_{\Sigma\Lambda}}^{A} \equiv {\vartheta_{\Sigma}}^{B}{T_{B\,\Lambda}}^{\Omega}{\vartheta_{\Omega}}^{A} - {\vartheta_{\Sigma}}^{B}{\vartheta_{\Lambda}}^{C}{f_{BC}}^{A}\,.$$

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which satisfy the algebra

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Then we construct the covariant 2-form field strengths

$$F^{\Lambda} = dA^{\Lambda} + \tfrac{1}{2}X_{\Sigma\Omega}{}^{\Lambda}A^{\Sigma}\wedge A^{\Omega}\,,$$

and the gauge -invariant action of the electrically gauged FT takes the form

$$S_{\rm eg}[Z^{i}, A^{\Lambda}] = \int \{-2\mathcal{G}_{ij^{*}}\mathfrak{D}Z^{i}\wedge\star\mathfrak{D}Z^{*\,j^{*}} - 2\Im\mathrm{m}f_{\Lambda\Sigma}F^{\Lambda}\wedge\star F^{\Sigma} + 2\Re\mathrm{e}f_{\Lambda\Sigma}F^{\Lambda}\wedge F^{\Sigma} - \star V_{\rm eg}(Z, Z^{*})\}$$

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$$dG_\Lambda = 0\,, \qquad \text{where} \qquad G_\Lambda{}^+ \equiv f_{\Lambda\Sigma} F^{\Sigma\,+}\,,$$

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- The theory (equations of motion) has more non-perturbative global symmetries that can be gauged . They include electric -magnetic duality rotations:

$$\begin{aligned}
\delta_\alpha Z^i &= \alpha^A k_A{}^i(Z)\,,\\
\delta_\alpha f_{\Lambda\Sigma} &= \alpha^A\{-T_{A\,\Lambda\Sigma} + 2T_{A\,(\Lambda}{}^{\Omega} f_{\Sigma)\Omega} - T_A{}^{\Omega\Gamma} f_{\Omega\Lambda} f_{\Gamma\Sigma}\}\,,\\
\delta_\alpha \begin{pmatrix} A^\Lambda \\ A_\Lambda \end{pmatrix} &= \alpha^A \begin{pmatrix} T_{A\,\Sigma}{}^{\Lambda} & T_A{}^{\Sigma\Lambda} \\ T_{A\,\Sigma\Lambda} & T_A{}^{\Sigma}{}_{\Lambda} \end{pmatrix} \begin{pmatrix} A^\Sigma \\ A_\Sigma \end{pmatrix}.
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Now we need to relate the α^A to the gauge parameters of the 1-forms Λ^Λ or Λ_Λ We need new (magnetic) components for the embedding tensor : $\vartheta^{\Lambda\, A}$. Then

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Knowing (Gaillard & Zumino) that the T_A matrices either belong to $\mathfrak{sp}(2n_V, \mathbb{R})$ or vanish, we introduce the symplectic notation

$$A^M \equiv \begin{pmatrix} A^\Sigma \\ A_\Sigma \end{pmatrix} \qquad \vartheta_M{}^A \equiv (\vartheta_\Sigma{}^A, \vartheta^{\Sigma\,A}) \qquad \alpha^A(x) \equiv \Lambda^M \vartheta_M{}^A\,,$$

$$(T_{A\,M}{}^N) \equiv \begin{pmatrix} T_{A\,\Sigma}{}^\Lambda & T_A{}^{\Sigma\Lambda} \\ T_{A\,\Sigma\Lambda} & T_A{}^\Sigma{}_\Lambda \end{pmatrix}\,.$$

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The electric and magnetic charges must by mutually local (de Wit, Samtleben & Trigiante, arXiv:`hep-th/0507289`):

$$Q^{AB} \equiv \tfrac{1}{4}\vartheta^{M\,A}\vartheta_M{}^B = 0\,.$$

Now we can repeat the procedure of the electric case:

First we construct derivatives $\mathfrak{D}$

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covariant under

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which only works if $\vartheta_M{}^A$ is an invariant tensor

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Before moving forward, we must impose another constraint on the embedding tensor on top of the two quadratic ones $Q_{MN}{}^A = Q^{AB} = 0$:

$$L_{MNP} \equiv X_{(MNP)} = \vartheta_{(M}{}^A T_{A\,NP)} = 0 .$$

This *linear* or *representation constraint* is based on supergravity and eliminates certain possible representations of the embedding tensor . On the other hand, we cannot construct gauge -covariant 2-form field strengths F^M without it!

4 – The 4-d tensor hierarchy

To construct the gauge -covariant 2-form field strengths F^{M} we take the covariant derivative of the gauge -covariant "field strength" $\mathcal{D}Z^{i}$:

$$\mathfrak{D}\mathfrak{D}Z^{i} = [dA^{M} + \tfrac{1}{2}X_{NP}{}^{M}A^{N}\wedge A^{P}]\vartheta_{M}{}^{A}k_{A}{}^{i}\,,$$

which suggests the definition

$$F^{M} \equiv dA^{M} + \tfrac{1}{2}X_{NP}{}^{M}A^{N}\wedge A^{P} + \Delta F^{M}\,, \qquad \vartheta_{M}{}^{A}\Delta F^{M} = 0\,,$$

so we have the Bianchi identity

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$\delta_{\Lambda}B_{A}$ is determined by the gauge -covariance of F^{M} plus $\delta B_{A} \sim d\Lambda_{A}$.

4 – The 4-d tensor hierarchy

To construct the gauge -covariant 2-form field strengths F^{M} we take the covariant derivative of the gauge -covariant "field strength" $\mathcal{D}Z^{i}$:

$$\mathfrak{D}\mathfrak{D}Z^{i} = [dA^{M} + \tfrac{1}{2}X_{NP}{}^{M}A^{N}\wedge A^{P}]\vartheta_{M}{}^{A}k_{A}{}^{i}\,,$$

which suggests the definition

$$F^{M} \equiv dA^{M} + \tfrac{1}{2}X_{NP}{}^{M}A^{N}\wedge A^{P} + \Delta F^{M}\,, \qquad \vartheta_{M}{}^{A}\Delta F^{M} = 0\,,$$

so we have the Bianchi identity

$$\boxed{\mathfrak{D}\mathfrak{D}Z^{i} = F^{M}\vartheta_{M}{}^{A}k_{A}{}^{i}\;.}$$

Using the constraint $Q^{AB} \equiv \frac{1}{4}\vartheta^{MA}\vartheta_{M}{}^{B} = 0$ the natural solution is

$$\Delta F^{M} = -\tfrac{1}{2}\vartheta^{MA}B_{A} \equiv Z^{MA}B_{A}\,.$$

$\delta_{\Lambda}B_{A}$ is determined by the gauge -covariance of F^{M} plus $\delta B_{A} \sim d\Lambda_{A}$.

But we do not need it to move forward.

If we take the covariant derivative of the gauge -covariant 2-form field strength F^{M} we find

$$\mathfrak{D}F^{M} = Z^{MA}\{\mathfrak{D}B_{A} + T_{A\,RS}A^{R}\wedge[dA^{S} + \tfrac{1}{3}X_{NP}{}^{S}A^{N}\wedge A^{P}]\}\,.$$

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The gauge -covariance of the l.h.s. suggests the definition

$$H_A = \mathfrak{D}B_A + T_{A\,RS}A^R \wedge [dA^S + \tfrac{1}{3}X_{NP}{}^S A^N \wedge A^P] + \Delta H_A\,, \qquad \text{where} \qquad Z^{MA}\Delta H_A = 0\,.$$

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Using the constraint

$$Q_{MN}{}^A = \vartheta_M{}^B(T_{B\,N}{}^P\vartheta_P{}^A - \vartheta_N{}^C f_{BC}{}^A) \equiv 2Z_M{}^A Y_{AN}{}^P = 0$$

the natural solution for $Z^{MA}\Delta H_A = Z^{MA}\Delta B_A = 0$ is

$$\Delta H_A \equiv Y_{AM}{}^C C_C{}^M\,.$$

$\delta_\Lambda C_C{}^M$ is fully determined by the gauge -covariance of H_A.

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But we do not need it to move forward.

If we take the covariant derivative of the gauge -covariant 3-form field strength H_A we find

$$\mathfrak{D}H_A - T_{A\,M\,N}F^M \wedge F^N = Y_{AM}{}^C\{\mathfrak{D}C_C{}^M + F^M \wedge B_C + \cdots\}\,.$$

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$$G_C{}^M = \mathfrak{D}C_C{}^M + F^M \wedge B_C + \cdots + \Delta G_C{}^M\,, \qquad \text{where} \qquad Y_{AM}{}^C \Delta G_C{}^M = 0\,.$$

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To determine $\Delta G_C{}^M$ we need to find invariant tensors that vanish upon contraction with $Y_{AM}{}^C$. They appear automatically when we take the gauge -covariant derivative of the Bianchi identity and $G_C{}^M$ (if we “forget” we are in 4 dimensions!).

Acting with $\mathfrak{D}$ on the Bianchi identity of H_A we find

$$Y_{AM}{}^{C}\{\mathfrak{D}G_{C}{}^{M} - F^{M} \wedge H_{A}\} = 0\,, \;\; \Rightarrow \;\; \mathfrak{D}G_{C}{}^{M} = F^{M} \wedge H_{A} + \Delta\mathfrak{D}G_{C}{}^{M}\,,$$

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Acting with $\mathfrak{D}$ on the above identity we find

$$\mathfrak{D}\Delta\mathfrak{D}G_{C}{}^{M} = W_{C}{}^{MAB}H_A\wedge H_B + W_{CNPQ}{}^{M}F^{N}\wedge F^{P}\wedge F^{Q} + W_{CNP}{}^{EM}F^{N}\wedge G_{E}{}^{P}\,.$$

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This implies that there are 3 such tensors $W_{C}{}^{MAB}, W_{CNPQ}{}^{M}, W_{CNP}{}^{EM}$ that vanish contracted with $Y_{AM}{}^{C}$ and which we can use to build $\Delta G_{C}{}^{M}$.

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The natural solution is

$$\Delta G_{C}{}^{M} = W_{C}{}^{MAB}D_{AB} + W_{CNPQ}{}^{M}D^{NPQ} + W_{CNP}{}^{EM}D_{E}{}^{NP}\,,$$

and $\delta_{\Lambda}D_{AB}, \delta_{\Lambda}D^{NPQ}, \delta_{\Lambda}D_{E}{}^{NP}$ will follow from the gauge -covariance of $G_{C}{}^{M}$.

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➠ A tower of $(p+1)$-forms $A^{M}, B_{A}, C_{C}{}^{M}, D_{AB}, D^{NPQ}, D_{E}{}^{NP}$ related by gauge transformations.

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\begin{aligned}
\delta_{\Lambda} A^{M} &= -\mathfrak{D}\Lambda^{M} - Z^{MA}\Lambda_{A}\,, \\
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\delta_{\Lambda} D_{AB} &= \mathfrak{D}\Lambda_{AB} + 2T_{[A\,MN}\tilde{\Lambda}_{B]}{}^{(MN)} + Y_{[A|P}{}^{E}(\Lambda_{B]E}{}^{P} - B_{B]} \wedge \Lambda_{E}{}^{P}) + \mathfrak{D}\Lambda_{[A} \wedge B_{B]} - 2\Lambda_{[A} \wedge H_{B]} \\
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&\quad + \Lambda^{N} G_{E}{}^{P}\,, \\
\delta_{\Lambda} D^{NPQ} &= \mathfrak{D}\Lambda^{NPQ} - 3Z^{(N|A}\tilde{\Lambda}_{A}{}^{|PQ)} - 2A^{(N} \wedge dA^{P} \wedge \delta_{\Lambda} A^{Q)} - \tfrac{3}{4} X_{RS}{}^{(N} A^{P|} \wedge A^{R} \wedge A^{S} \wedge \delta_{\Lambda} A^{|Q)} - 3\Lambda^{(N} F^{P}
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This system is known as the (4-dimensional) *tensor hierarchy*.

It is *universal*: it exists for all 4-dimensional FTs with gauge symmetry.

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But, what does it mean?
What is the meaning of the additional fields?

5 – The meaning of the $d = 4$ tensor hierarchy

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- ☞ These two duality relations together with the Bianchi identity $\mathfrak{D}F^M = Z^{MA} H_A$ give a set of electric -magnetic duality -covariant Maxwell equations:

$$\mathfrak{D}F^\Lambda = -\tfrac{1}{4}\vartheta_\Lambda{}^A \star j_A \ , \qquad \mathfrak{D}G_\Lambda = \tfrac{1}{4}\vartheta^{\Lambda\,A} \star j_A \ .$$

☞ The 3-forms $C_{C}{}^{M}$ must be dual to constants: the embedding tensor $\vartheta_{M}{}^{C}$. This duality is expressed through the formula

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\mathfrak{D} \star j_{A} = 4 T_{A\,MN} G^{M} \wedge G^{N} + \star {Y_{A}}^{MC} \frac{\partial V}{\partial {\vartheta_{M}}^{C}} \; .
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$$

This equation is similar to the consistency condition (gauge or Noether identity) that Noether currents must satisfy off-shell in FTs with gauge invariance:

$$
\mathfrak{D} \star j_A = -2({k_A}^i \mathcal{E}_i + \text{c.c.}) + 4 T_{A\,MN} G^M \wedge G^N + \star {Y_A}^{MC} \frac{\partial V}{\partial {\vartheta_M}^C} \,,
$$

where $\mathcal{E}_i$ is the e.o.m. of Z^i. Both equations, together, imply

$$
{k_A}^i \mathcal{E}_i + \text{c.c.} = 0 \quad ,
$$

which is equivalent to the scalar e.o.m. for symmetric σ-models.

- ☞ Finally, the indices of the 3 4-forms $D_{AB}, D^{NPQ}, D_{E}{}^{NP}$ are conjugate to those of the constraints $Q^{AB}, Q_{NPQ}, Q_{NP}{}^{E}$. They are Lagrange multipliers enforcing them.

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To confirm this interpretation we must construct a gauge -invariant action for **all** these fields, including the embedding tensor $\vartheta_M{}^A(x)$.

This gauge -invariant action is given by

$$
\begin{aligned}
S[g_{\mu\nu}, Z^i, A^M, B_A, C_A{}^M, D_E{}^{NP}, D_{AB}, D^{MNP}, \vartheta_M{}^A] \quad = \\
\int \big\{ -2\mathcal{G}_{ij^*}\mathfrak{D}Z^i \wedge \star\mathfrak{D}Z^{*\,j^*} + 2F^\Sigma \wedge G_\Sigma - \star V \\
-4Z^{\Sigma A} B_A \wedge \left(F_\Sigma - \tfrac{1}{2}Z_\Sigma{}^B B_B\right) - \tfrac{4}{3}X_{[MN]\Sigma}A^M \wedge A^N \wedge \left(F^\Sigma - Z^{\Sigma B}B_B\right) \\
-\tfrac{2}{3}X_{[MN]}{}^\Sigma A^M \wedge A^N \wedge \left(dA_\Sigma - \tfrac{1}{4}X_{[PQ]\Sigma}A^P \wedge A^Q\right) \\
-2\mathfrak{D}\vartheta_M{}^A \wedge (C_A{}^M + A^M \wedge B_A) \\
+2Q_{NP}{}^E(D_E{}^{NP} - \tfrac{1}{2}A^N \wedge A^P \wedge B_E) + 2Q^{AB}D_{AB} + 2L_{MNP}D^{MNP} \big\} \,.
\end{aligned}
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---

6 – Application: general gaugings of $N = 1, d = 4$ supergravity

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We are going to review ungauged $N = 1$ supergravity and its global symmetries and then we are going to gauge them using the embedding tensor formalism.

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All fermions are represented by chiral 4-component spinors:

$$\gamma_5 \psi_{\mu} = -\psi_{\mu}\,, \text{etc.}$$

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The spinors transform as *sections* of the bundle: under Kähler transformations

$$\delta_\lambda \mathcal{K} = \lambda(Z) + \lambda^*(Z^*)\,, \qquad \delta_\lambda \psi_\mu = -\tfrac{1}{4}[\lambda(Z) - \lambda^*(Z^*)]\psi_\mu\,,$$

and their covariant derivatives contain the pullback of the Kähler connection 1-form $\mathcal{Q} \equiv \mathcal{Q}_i dZ^i + \mathcal{Q}_{i^*}dZ^{*i^*}$ e.g.

$$\mathcal{D}_\mu \psi_\nu = \{\nabla_\mu + \tfrac{i}{2}\mathcal{Q}_\mu\}\psi_\nu\,.$$

$N = 1$ supergravity allows for an arbitrary holomorphic kinetic matrix $f_{\Lambda\Sigma}(Z)$ for the vector fields which occurs in the action in the terms

$$-2\Im\mathrm{m} f_{\Lambda\Sigma} F^{\Lambda} \wedge \star F^{\Sigma} + 2\Re\mathrm{e} f_{\Lambda\Sigma} F^{\Lambda} \wedge F^{\Sigma} \,, \qquad\qquad F^{\Lambda} \equiv dA^{\Lambda} \,.$$

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Finally, ungauged $N=1$ supergravity allows for a holomorphic superpotential $\mathcal{W}(Z)$ which appears through the covariantly holomorphic section of Kähler weight $(1,-1)$ $\mathcal{L}(Z,Z^*)$:

$$\mathcal{L}(Z,Z^*) = \mathcal{W}(Z) e^{\mathcal{K}/2} \, , \qquad \mathcal{D}_{i^*}\mathcal{L} = 0 \, ,$$

which couples to the fermions in various ways and gives rise to the scalar potential

$$V_{\mathrm{u}}(Z,Z^*) = -24|\mathcal{L}|^2 + 8\mathcal{G}^{ij^*}\mathcal{D}_i\mathcal{L}\mathcal{D}_{j^*}\mathcal{L}^* \, .$$

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The bosonic action is

$$S_{\mathrm{u}}[g_{\mu\nu}, Z^i, A^{\Lambda}] = \int \{\star R - 2\mathcal{G}_{ij^*} dZ^i \wedge \star dZ^{*\,j^*} - 2\Im\mathrm{m} f_{\Lambda\Sigma} F^{\Lambda} \wedge \star F^{\Sigma}$$

$$+2\Re\mathrm{e} f_{\Lambda\Sigma} F^{\Lambda} \wedge F^{\Sigma} - \star V_{\mathrm{u}}(Z, Z^*)\}\,.$$

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- ☞ But this would mean that we are dealing with a $A = \text{a}$ symmetry and we can say that a non-vanishing superpotential breaks $U(1)_R$ and we cannot gauge it.

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- According to the previous discussion, the symmetries $A = \underline{a}, \#$ are broken and cannot be gauged if $\mathcal{L} \neq 0$.

$$\mathcal{L} \neq 0\,,\ \Rightarrow \vartheta_M{}^A(\delta_A{}^{\underline{a}}\mathcal{P}_{\underline{a}} + \delta_A{}^{\#}\mathcal{P}_{\#}) = 0\ .$$

9 – The $N = 1, d = 4$ bosonic tensor hierarchy

We have found that, for non-vanishing superpotential , the embedding tensor must satisfy another constraint

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$$\Delta' H_A \equiv \Delta H_A + Y_A C\,, \qquad Y_A \equiv (\delta_A{}^{\underline{a}}\mathcal{P}_{\underline{a}} + \delta_A{}^{\#}\mathcal{P}_{\#})\,.$$

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This will happen in $N=1$ supergravity if we find new Stückelberg shifts

$$\delta' B_A \sim \delta_h B_A + Y_A \Lambda \quad \text{and} \quad \delta' C_C{}^M = \delta_h C_C{}^M + Y_C \Lambda^M\,.$$

10 – The $N = 1, d = 4$ supersymmetric tensor hierarchy

As a first step to include the tensor hierarchy fields into $N = 1$ supergravity we are going to construct supersymmetry transformation rules such that the local supersymmetry algebra, to leading order in fermions , closes on the new fields up to duality relations.

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Observe that we are going to obtain, independently, the gauge transformations of the fields and will confirm or refute the hierarchy's results.

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$$\delta_\eta \chi^i = i\,\not{\mathfrak{D}} Z^i \eta^* + 2\mathcal{G}^{ij^*}\mathcal{D}_{j^*}\mathcal{L}^*\eta\,, \qquad \mathfrak{D}Z^i = dZ^i + A^M \vartheta_M{}^A k_A{}^i\,.$$

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We find the expected result

$$\begin{aligned}
[\delta_\eta\,,\,\delta_\epsilon]Z^i &= \delta_{\text{g.c.t.}}Z^i + \delta_h Z^i\,,\\
\delta_{\text{g.c.t.}}Z^i &= \pounds_\xi Z^i = +\xi^\mu\partial_\mu Z^i\,,\\
\delta_h Z^i &= \Lambda^M\vartheta_M{}^A k_A{}^i\,,\\
\xi^\mu &\equiv \tfrac{i}{4}(\bar{\epsilon}\gamma^\mu\eta^* - \bar{\eta}\gamma^\mu\epsilon^*)\,,\\
\Lambda^M &\equiv \xi^\mu A^M{}_\mu\,.
\end{aligned}$$

The 1-forms A^M

We introduce supersymmetric partners λ_Σ for the magnetic 1-forms A_Σ and make the symplectic -covariant Ansatz

$$
\begin{aligned}
\delta_\epsilon A^M{}_\mu &= -\tfrac{i}{8}\bar{\epsilon}^*\gamma_\mu\lambda^M + \text{c.c.}\,, \\
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where we have defined the symplectic vector

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\mathcal{D}^M \equiv \begin{pmatrix} \mathcal{D}^\Lambda \\ \mathcal{D}_\Lambda \end{pmatrix} \equiv \begin{pmatrix} \mathcal{D}_\Lambda \\ f_{\Lambda\Sigma}\mathcal{D}^\Sigma \end{pmatrix}, \qquad \mathcal{D}^\Lambda = -\Im\text{m} f^{\Lambda\Sigma}\,(\vartheta_\Sigma{}^A + f^*_{\Sigma\Omega}\vartheta^{\Omega\,A})\mathcal{P}_A\,.
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The 2-forms B_A

We introduce the supersymmetric partners ζ_A, φ_A (linear supermultiplets)

$$\begin{aligned}
\delta_\epsilon \zeta_A &= -i[\tfrac{1}{12} \not{H}'_A + \not{\mathfrak{D}}\varphi_A]\epsilon^* - 4\delta_A{}^{\mathbf{a}}\varphi_{\mathbf{a}}\mathcal{L}^*\epsilon\,, \\
\delta_\epsilon B_{A\,\mu\nu} &= \tfrac{1}{4}[\bar{\epsilon}\gamma_{\mu\nu}\zeta_A + \text{c.c.}] - i[\varphi_A\bar{\epsilon}^*\gamma_{[\mu}\psi_{\nu]} - \text{c.c.}] + 2T_{A\,MN}A^M{}_{[\mu}\delta_\epsilon A^N{}_{\nu]}\,, \\
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which **proofs** the existence of an extra Stückelberg shift in B_A.

The 3-forms $C_A{}^M$

In this case we do not introduce any supersymmetric partners. We just make the Ansatz

$$\delta_\epsilon C_A{}^M{}_{\mu\nu\rho} = -\tfrac{i}{8}[\mathcal{P}_A \bar{\epsilon}^* \gamma_{\mu\nu\rho} \lambda^M - \text{c.c.}] - 3B_{A\,[\mu\nu|}\delta_\epsilon A^M{}_{|\rho]} - 2T_{A\,PQ} A^M{}_{[\mu} A^P{}_{\nu|} \delta_\epsilon A^Q{}_{|\rho]}\,.$$

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This corresponds to a scalar potential of the form

$$V_{\mathrm{e-mg}} = V_\mathrm{u} - \tfrac{1}{2}\Re\mathrm{e}\, \mathcal{D}^M \vartheta_M{}^A \mathcal{P}_A = V_\mathrm{u} + \tfrac{1}{2}\mathcal{M}^{MN} \vartheta_M{}^A \vartheta_N{}^B \mathcal{P}_A \mathcal{P}_B\,,$$

where

$$(\mathcal{M}^{MN}) \equiv \begin{pmatrix} I^{\Lambda\Sigma} & I^{\Lambda\Omega} R_{\Omega\Sigma} \\ R_{\Lambda\Omega} I^{\Omega\Sigma} & I_{\Lambda\Sigma} + R_{\Lambda\Omega} I^{\Omega\Gamma} R_{\Gamma\Sigma} \end{pmatrix}, \qquad \begin{array}{rcl} f_{\Lambda\Sigma} & \equiv & R_{\Lambda\Sigma} + i I_{\Lambda\Sigma}\,, \\ I^{\Lambda\Omega} I_{\Omega\Sigma} & \equiv & \delta^\Lambda{}_\Sigma\,, \end{array}$$

so it is manifestly symplectic -invariant, as it must.

The 3-forms C, C'

The consistency of the previous results requires the existence of a 3-form C transforming under the extra Stückelberg shift of B_A.

$$\delta_\epsilon C_{\mu\nu\rho} = -3i\mathbf{g}\mathcal{L}\,\bar{\epsilon}^{*}\gamma_{[\mu\nu}\psi^{*}{}_{\rho]} - \tfrac{1}{2}\mathbf{g}\mathcal{D}_i\mathcal{L}\bar{\epsilon}^{*}\gamma_{\mu\nu\rho}\chi^{i} + \text{c.c.}\,,$$

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$$G' = \star\mathbf{g}(-24|\mathcal{L}|^2 + 8\mathcal{G}^{ij^*}\mathcal{D}_i\mathcal{L}\mathcal{D}_{j^*}\mathcal{L}^*)\,.$$

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If we rescale the superpotential by $\mathcal{L} \to \mathbf{g}\mathcal{L}$, the above duality relation takes the standard form

$$G' = \tfrac{1}{2}\star\frac{\partial V_{\text{e}-\text{mg}}}{\partial\mathbf{g}}\,,$$

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So, what is the 3-form C dual to?

The 4-forms $D_{AB}, D^{NPQ}, D_E{}^{NP}, D^M$

The calculations become horribly complicated and we only check the closure of the local supersymmetry algebra in the ungauged $\vartheta_M{}^A = 0$ case when there are no symmetries acting on the 1-forms i.e. $T_{A\,M}{}^N = 0$.

The supersymmetry transformations are

$$\begin{aligned}
\delta_\epsilon D_{AB} &= -\tfrac{i}{2} \star \mathcal{P}_{[A}\partial_i \mathcal{P}_{B]}\bar{\epsilon}\chi^i + \text{c.c.} - B_{[A} \wedge \delta_\epsilon B_{B]}\,, \\
\delta_\epsilon D^{NPQ} &= 10 A^{(N} \wedge F^P \wedge \delta_\epsilon A^{Q)}\,, \\
\delta_\epsilon D_E{}^{NP} &= C_E{}^P \wedge \delta_\epsilon A^N\,. \\
\delta_\epsilon D^M &= -\tfrac{i}{2} \star \mathcal{L}^* \bar{\epsilon}\lambda^M + \text{c.c.} + C \wedge \delta_\epsilon A^M\,.
\end{aligned}$$

This proves that D^M can be consistently added to the supersymmetric theory. Its role in the action will be that of Lagrange multiplier of the constraint Q_M.

11 – The supersymmetric objects of $N = 1$ supergravity

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pp-waves ($e^a{}_\mu$), strings (B_A) and domain walls (C, C').

We are going to focus on the domain walls associated to the 3-form C' since we need to know the associated deformation parameter in order to couple C to supergravity . We consider the ungauged theory with only chiral supermultiplets and superpotential

12 – Domain-wall solutions of $N = 1$ supergravity

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The metric of a 4-d domain-wall solution can always be written in the form

$$ds^2 = H\eta_{\mu\nu}dx^\mu dx^\nu = H(y)[\eta_{mn}dx^m dx^n - dy^2]\,, \qquad m,n = 0,1,2\,.$$

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If the $Z^i = Z^i(y)$ the gravitino Killing spinor equation $\delta_\epsilon \psi_\mu = 0$ is be solved by

$$(e^{-i\alpha/2}\epsilon) \pm i\gamma^{012}(e^{-i\alpha/2}\epsilon)^* = 0\,, \qquad e^{i\alpha} \equiv \mathcal{L}/|\mathcal{L}|\,.$$

and $H(y)$ satisfies the "*H flow equation*"

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These two first-order *flow equations* imply the second-order supergravity equations of motion.

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where $\beta \in \mathbb{R}$, $|g_3|$ the determinant of the pullback $g_{(3)\,mn}$ of the spacetime metric over the 3-dimensional worldvolume and C'_{mnp} is the pullback of the 3-form.

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In the static gauge $\partial X^\mu/\partial\xi^m = \delta^\mu{}_m$ it can be seen that this action is invariant to lowest order in fermions under the supersymmetry transformations of $g_{\mu\nu}, Z^i, C'_{\mu\nu\rho}$ if $\beta = \pm 1/4$ and the spinors satisfy the BPS domain-wall projection $(e^{-i\alpha/2}\epsilon) \pm i\gamma^{012}(e^{-i\alpha/2}\epsilon)^* = 0.$

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Thus, we consider the bulk supergravity action,

$$S_{\rm bulk} = \frac{1}{\kappa^2}\int d^4x\sqrt{|g|}\left[R + 2\mathcal{G}_{ij^*}\partial_\mu Z^i\partial^\mu Z^{*\,j^*} - \mathbf{g}^2(x)V(Z,Z^*) - \frac{1}{3\sqrt{|g|}}\epsilon^{\mu\nu\rho\sigma}\partial_\mu\mathbf{g}(x)C'_{\nu\rho\sigma}\right]$$

14 – Sourceful domain-wall solutions of $N = 1$ supergravity

To couple our candidate to domain-wall source to the bulk $N = 1$ supergravity action we need to introduce consistently C' into it by

1. Rescaling $\mathcal{L} \longrightarrow \mathbf{g}\mathcal{L}$ everywhere except at the 3-form supersymmetry transformation rule.
2. Promoting the constant $\mathbf{g}$ to a scalar field $\mathbf{g}(x)$ and adding to the bulk supergravity action a Lagrange -multiplier term containing the 3-form to enforce the constancy of $\mathbf{g}$.

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and the brane source action

$$S_{\text{brane}} = -\int d^4x \mathbf{f}(y)\left\{|\mathcal{L}|\sqrt{|g_{(3)}|} \pm \frac{1}{4!}\epsilon^{mnp}C'_{\underline{mnp}}\right\},$$

where $\mathbf{f}(y)$ is a distribution function of domain walls common transverse direction $x^3 \equiv y$: $\mathbf{f}(y) = \delta^{(1)}(y - y_0)$ for a single domain wall placed at $y = y_0$ etc.

The equations of motion that follow from $S \equiv S_{\rm bulk} + S_{\rm brane}$ are

$$\begin{aligned}
\mathcal{E}_{\mathbf{g}}^{\mu\nu} &= -\tfrac{\kappa^2}{2}\mathbf{f}(y)|\mathcal{L}|\frac{\sqrt{|g_{(3)}|}}{\sqrt{|g|}} g_{(3)}^{mn}\delta_m{}^{\mu}\delta_n{}^{\nu}\,, \\
\mathcal{G}^{ij^*}\mathcal{E}_{\mathbf{g}\,i^*} &= -\tfrac{\kappa^2}{8}\mathbf{f}(y)\frac{\sqrt{|g_{(3)}|}}{\sqrt{|g|}} e^{i\alpha}\mathcal{N}^i\,, \\
\epsilon^{\mu\nu\rho\sigma}\partial_\sigma \mathbf{g}(x) &= \pm\tfrac{\kappa^2}{8}\mathbf{f}(y)\epsilon^{mnp}\delta_m{}^{\mu}\delta_n{}^{\nu}\delta_p{}^{\rho}\,, \\
\epsilon^{\mu\nu\rho\sigma}\partial_\mu C'_{\nu\rho\sigma} &= 6\mathbf{g}(x)V(Z,Z^*)\,,
\end{aligned}$$

where $\mathcal{E}_{\mathbf{g}}^{\mu\nu}$ and $\mathcal{E}_{\mathbf{g}\,i^*}$ are the Einstein and scalar equations of motion with $\mathbf{g}(x) \neq 0$.

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The third equation is that of the 3-form and is solved if $\mathbf{g}$ is a function of y satisfying

$$\partial_{\underline{y}}\mathbf{g} = \pm\frac{\kappa^2}{8}\mathbf{f}(y)\,.$$

$\mathbf{g}(y)$ will have step-like discontinuities at the locations of the domain walls.

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The fourth equation is that of $\mathbf{g}(x)$ and y states that C' is the dual of the scalar potential.

It can now be checked that the Einstein and scalar equations of motion are identically satisfied if $H(y)$ and the scalars $Z^i(y)$ satisfy the *sourceful flow equations*

$$\partial_{\underline{y}} Z^i = \pm \mathbf{g}(y) e^{i\alpha} \mathcal{N}^i H^{1/2}\,,$$

$$\partial_{\underline{y}} H^{-1/2} = \pm 2\mathbf{g}(y)|\mathcal{L}|\,.$$

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These equations can be derived from the modified fermion supersymmetry transformations

$$\begin{aligned} \delta_\epsilon \psi_\mu &= \mathcal{D}_\mu \epsilon + i\mathbf{g}(x)\mathcal{L}\gamma_\mu \epsilon^*\,, \\ \delta_\epsilon \chi^i &= i\,\not{\partial} Z^i \epsilon^* + \mathbf{g}(x)\mathcal{N}^i \epsilon\,. \end{aligned}$$

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A fully supersymmetric "democratic" formulation of $N = 1$ $d = 4$ supergravity including all higher-rank forms and local coupling constants is necessary to accomodate these modifications.

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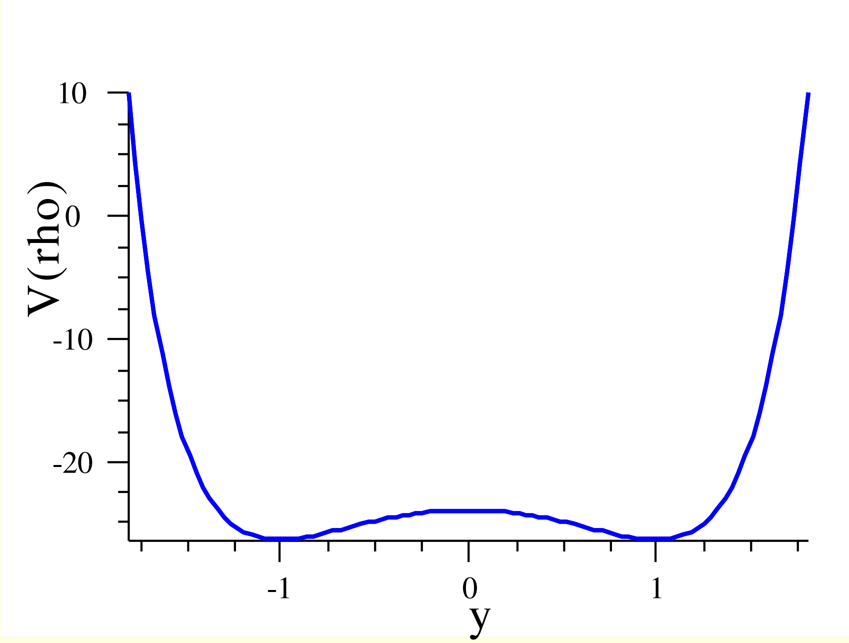
Observe that the space-dependent coupling constant $\mathbf{g}(x)$, sourced by domain wall , can modify the effective scalar potential dramatically.

15 – A simple example

Let us consider the model (1 chiral multiplet) defined by

$$\mathcal{K} = |Z|^2\,, \qquad \mathcal{W} = 1\,, \qquad (\mathcal{L} = e^{|Z|^2/2}\,,\ \mathcal{N}^Z = 2Z^* e^{|Z|^2/2})\,.$$

These choices lead to the Mexican-hat-type potential $V = -8(3-\rho^2)e^{\rho^2/2}$ ($Z \equiv \rho e^{i\beta}$) with a maximum and degenerate minimum at $\rho = 0$ and $\rho = +1$ resp. with $V(0) = -24\,,\ V(1) = -16\sqrt{e} \sim -26.4$.



These numbers are irrelevant, as V will be multiplied by $\mathbf{g}^2(y)$, determined by the sources .

The *sourceful flow equations* take the form ($\beta = \text{Arg}\, Z = \text{const}$)

$$\begin{aligned} \partial_{\underline{y}}\rho &= \pm 2\mathbf{g}(y)\rho e^{\rho^2/2} H^{1/2}\,, \\ \partial_{\underline{y}} H^{-1/2} &= \pm 2\mathbf{g}(y) e^{\rho^2/2}\,. \end{aligned}$$

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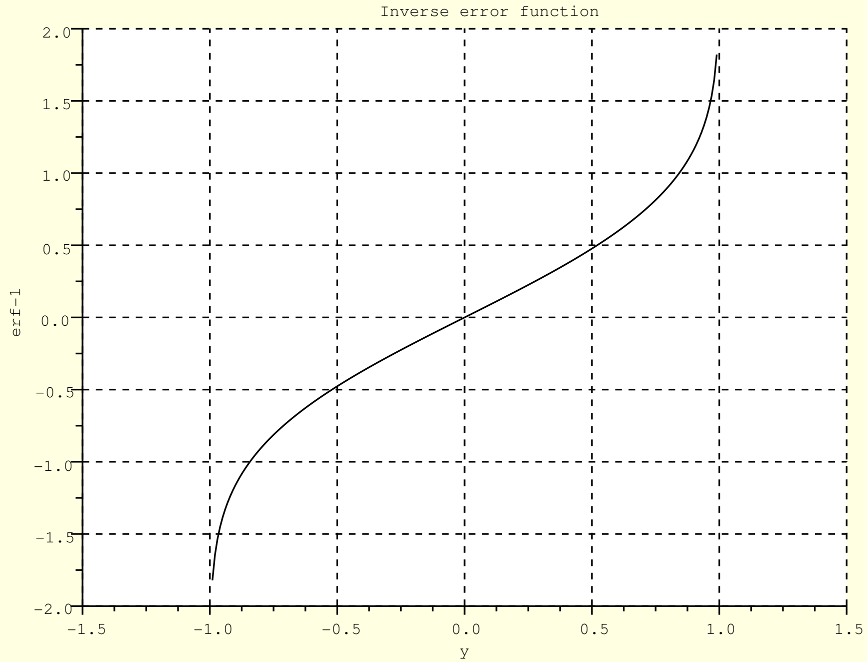
II-b Solutions with $\mathbf{g} \neq 0$ and $\partial_{\underline{y}} Z \neq 0$:

$$H = c/\rho^2\,,$$

$$\rho = \sqrt{2}\,\text{erf}^{-1}\left[\mathbf{G}(y)\right]\,, \qquad \mathbf{G}(y) \equiv \pm\sqrt{\tfrac{8c}{\pi}} \int \mathbf{g}(y)dy + d\,.$$

erf^{-1} is the inverse of the *normalized error function*

$$\mathrm{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du = -\mathrm{erf}(-x)\,.$$



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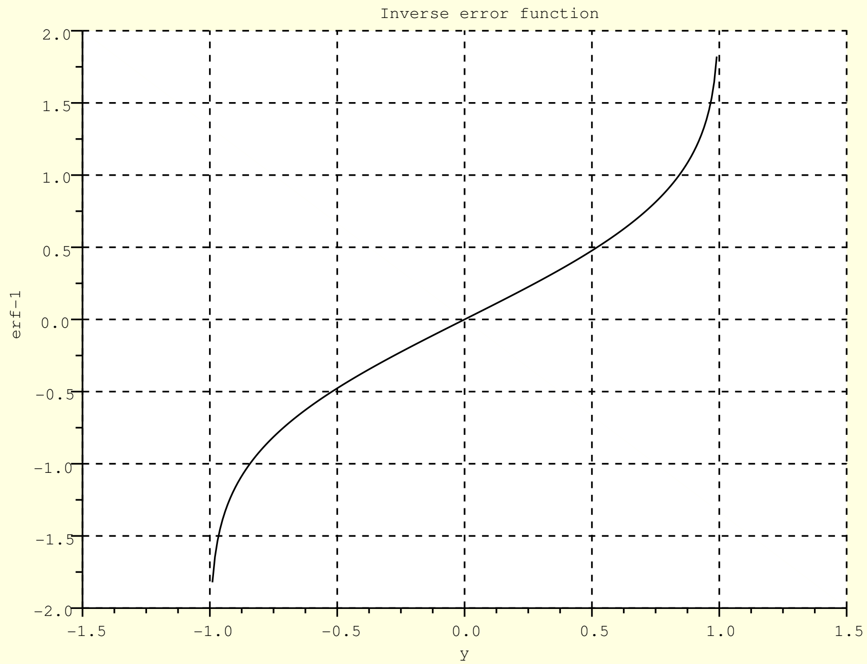


Then, for constant $\mathbf{g}$, the solution is restricted to $y \in (-1, +1)$ and it is locally AdS_4, but cannot be interpreted as an interpolation between two vacua unless we cut the spacetime at finite values of y. To have more general $\mathbf{g}(y)$ we have to add sources.

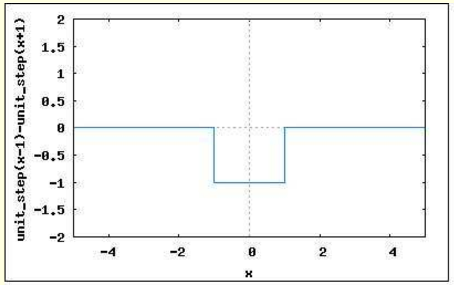
Let us consider, first, a single, infinitely thin domain-wall source of tension $q > 0$ placed at $y = y_0$:

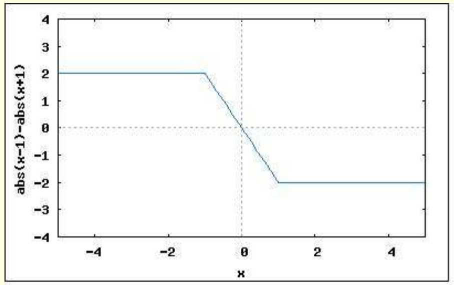
$$\mathbf{f}(y) = q\delta(y - y_0)\,, \qquad \mathbf{g}(y) = \pm\frac{\kappa^2 q}{16}[\theta(y-y_0) - \theta(y_0 - y)]\,, \qquad \mathbf{G}(y) = \frac{\sqrt{c}\kappa^2 q}{\sqrt{32\pi}}|y - y_0| + d\,.$$

$\mathbf{G}(y)$ is always unbounded and the solution is not well defined.

A possible solution: we introduce two parallel domain wall with opposite tension and charge at a different point ($y = -y_0$ with $y_0 > 0$ for simplicity) so

$$\begin{aligned}
\mathbf{f}(y) &= q\delta(y - y_0) - q\delta(y + y_0)\,,\\
\mathbf{g}(y) &= \pm\frac{\kappa^2 q}{16}[\theta(y - y_0) - \theta(y_0 - y) - \theta(y + y_0) + \theta(-y_0 - y)]\,,\\
\mathbf{G}(y) &= \sqrt{\frac{c}{32\pi}}\kappa^2 q\,(|y - y_0| - |y + y_0|) + d\,.
\end{aligned}$$





Choosing $d = \sqrt{\frac{c}{8\pi}}\kappa^2 q y_0$ we can set $\mathbf{G}(+\infty) = \mathbf{G}(+y_0) = 0$ and $\rho(y) = \rho(+y_0) = 0 \quad y > y_0$.

In the interior of the $\mathbf{g}(y) \neq 0$ region ρ approaches zero as $\rho \sim \frac{1}{4}\sqrt{c}\kappa^2 q(y_0 - y)$ so the metric approaches AdS_4

$$H \sim \frac{R^2}{(y_0 - y)^2}\,, \qquad\qquad R = \frac{4}{\kappa^2 q}\,.$$

The value $\mathbf{G}(-y_0) = \sqrt{\frac{c}{2\pi}}\kappa^2 q y_0 = \mathbf{G}(-\infty)$, can be tuned by moving the domain-wall sources (y_0). It has to be smaller or equal than 1.

If $\mathbf{G}(-y_0) < 1$ then $\rho(-y_0)$ is finite and ρ approaches $y = -y_0$ from the interior of the $\mathbf{g}(y) \neq 0$ region as

$$\rho \sim -\sqrt{\frac{c}{2\pi}}\frac{\kappa^2 q}{\mathrm{erf}'[\rho(-\infty)/\sqrt{2}]}(y + y_0)\,,$$

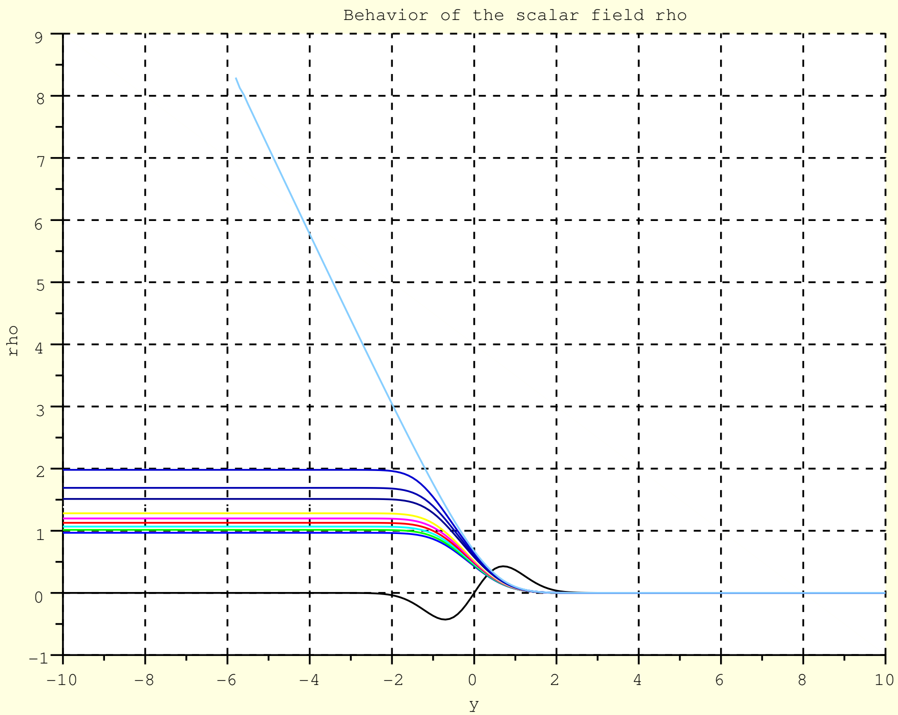
so the metric approaches another AdS_4 region.

This solution we have obtained smoothly interpolates between two AdS_4 regions one of which (the $\rho = 0$ one) corresponds to a supersymmetric vacuum of the theory.

The two infinitely-thin domain-wall sources setup can be understood as a crude approximation to the following configuration with domain-wall sources of finite thickness

$$\mathbf{f}(y) = qye^{-y^2}\,, \qquad \mathbf{g}(y) = \mp\frac{\kappa^2 q}{16}e^{-y^2}\,, \qquad \mathbf{G}(y) = -\frac{\kappa^2 q\sqrt{c}}{8}\mathrm{erf}(y) + d\,.$$

in which $\mathbf{g}(y)$ only vanishes asymptotically.



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- ★ We have seen that in some cases domain-wall sources have to be introduced to construct sensible solutions. These sources introduce a spacetime-dependent coupling constant $\mathbf{g}(x)$ that can have dramatic effects.