The tensor hierarchy and supersymmetric solutions of N=1,d=4 gauged supergravity

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Talk given on the 12th of October 2009 at the *Center for Theoretical Physics, University of Groningen, The Netherlands*

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Work done in collaboration with *E. Bergshoeff, O. Hohm* (U. Groningen) *J. Hartong* (U. Bern) *M. Hübscher* and *P. Meessen* (IFT UAM/CSIC, Madrid)

Plan of the Talk:

1 – Introduction/motivation

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3. The *embedding tensor method* (Cordaro, Fré, Gualtieri,Termonia & Trigiante, arXiv:hep-th/9804056.) can be used to construct systematically the most general gauged supergravities . This construction requires the introduction of additional $(p+1)$-form potentials.

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We are going to use the embedding tensor method to find all the $(p+1)$-form potentials and the corresponding democratic formulations of any 4-dimensional FT with gauge symmetry and we are going to apply the general results to the particular case of $N=1$ supergravity .

What we are going to do in this seminar:

[a]So far, only maximal and half-maximal supergravities have been studied from this point of view de Wit, Samtleben & Trigiante, arXiv:hep-th/0412173, Samtleben & Weidner arXiv:hep-th/0506237, Schon & Weidner, arXiv:hep-th/0602024, de Wit, Samtleben & Trigiante, arXiv:0705.2101, Bergshoeff, Gomis, Nutma & Roest, arXiv:0711.2035, de Wit, Nicolai & Samtleben, arXiv:0801.1294. The only exception is de Vroome & de Wit arXiv:0707.2717, but the $U(2)$ R-symmetry group has not been properly taken into account.

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5. We will use some of the new $(p + 1)$-form potentials to construct supersymmetric p-brane effective actions and solutions with sources of $N = 1$ supergravity .

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2 – The embedding tensor method: electric gaugings

Consider a general ($N = 1$ supergravity -inspired) 4-dimensional ungauged FT with bosonic fields $\{Z^i, A^\Lambda\}$ (the metric plays no relevant role here)

$$S_{\rm u}[Z^i, A^\Lambda] = \int \{-2\mathcal{G}_{ij^*} dZ^i \wedge \star dZ^{*\,j^*} - 2\Im{\rm m} f_{\Lambda\Sigma} F^\Lambda \wedge \star F^\Sigma + 2\Re{\rm e} f_{\Lambda\Sigma} F^\Lambda \wedge F^\Sigma - \star V_{\rm u}(Z, Z^*)\}\,.$$

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The action is invariant under the local Abelian transformations

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Let us assume this action is invariant under the global transformations

$$\begin{aligned}
\delta_\alpha Z^i &= \alpha^A k_A{}^i(Z)\,,\\
\delta_\alpha f_{\Lambda\Sigma} &\equiv -\alpha^A \pounds_A f_{\Lambda\Sigma} = \alpha^A[T_{A\,\Lambda\Sigma} - 2T_{A\,(\Lambda}{}^\Omega f_{\Sigma)\Omega}]\,,\\
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Each embedding tensor $\vartheta_{\Lambda}{}^{A}$ defines a possible identification:

$$\alpha^{A}(x) \equiv \Lambda^{\Sigma}\vartheta_{\Sigma}{}^{A}\,, \qquad\qquad A^{A} \equiv A^{\Sigma}\vartheta_{\Sigma}{}^{A}\,.$$

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Now we construct derivatives $\mathfrak{D}$

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covariant under

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This only works if ${\vartheta_\Lambda}^A$ is an invariant tensor

$$\delta_\Lambda {\vartheta_\Sigma}^A = -\Lambda^\Omega Q_{\Omega\Sigma}{}^A = 0\,, \qquad Q_{\Sigma\Lambda}{}^A \equiv {\vartheta_\Sigma}^B T_{B\,\Lambda}{}^\Omega {\vartheta_\Omega}^A - {\vartheta_\Sigma}^B {\vartheta_\Lambda}^C f_{BC}{}^A\,.$$

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which satisfy the algebra

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Then we construct the covariant 2-form field strengths

$$F^\Lambda = dA^\Lambda + \tfrac{1}{2} X_{\Sigma\Omega}{}^\Lambda A^\Sigma \wedge A^\Omega\,,$$

and the gauge -invariant action of the electrically gauged FT takes the form

$$S_{\rm eg}[Z^i, A^\Lambda] = \int \{-2\mathcal{G}_{ij^*}\mathfrak{D}Z^i\wedge\star\mathfrak{D}Z^{*\,j^*} - 2\Im\mathrm{m} f_{\Lambda\Sigma}F^\Lambda\wedge\star F^\Sigma + 2\Re\mathrm{e} f_{\Lambda\Sigma}F^\Lambda\wedge F^\Sigma - \star V_{\rm eg}(Z, Z^*)\}$$

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- The theory (equations of motion) has more non-perturbative global symmetries that can be gauged . They include electric -magnetic duality rotations:

$$\begin{aligned}
\delta_\alpha Z^i &= \alpha^A k_A{}^i(Z)\,,\\
\delta_\alpha f_{\Lambda\Sigma} &= \alpha^A\{-T_{A\,\Lambda\Sigma} + 2T_{A\,(\Lambda}{}^{\Omega} f_{\Sigma)\Omega} - T_A{}^{\Omega\Gamma} f_{\Omega\Lambda} f_{\Gamma\Sigma}\}\,,\\
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Now we need to relate the α^A to the gauge parameters of the 1-forms Λ^Λ or Λ_Λ We need new (magnetic) components for the embedding tensor : $\vartheta^{\Lambda\, A}$. Then

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Knowing (Gaillard & Zumino) that the T_A matrices either belong to $\mathfrak{sp}(2n_V, \mathbb{R})$ or vanish, we introduce the symplectic notation

$$A^M \equiv \begin{pmatrix} A^\Sigma \\ A_\Sigma \end{pmatrix} \qquad \vartheta_M{}^A \equiv (\vartheta_\Sigma{}^A, \vartheta^{\Sigma\, A}) \qquad \alpha^A(x) \equiv \Lambda^M \vartheta_M{}^A\,,$$

$$(T_{A\, M}{}^N) \equiv \begin{pmatrix} T_{A\,\Sigma}{}^\Lambda & T_A{}^{\Sigma\Lambda} \\ T_{A\,\Sigma\Lambda} & T_A{}^\Sigma{}_\Lambda \end{pmatrix}\,.$$

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The electric and magnetic charges must by mutually local (de Wit, Samtleben & Trigiante, arXiv:`hep-th/0507289`):

$$Q^{AB} \equiv \tfrac{1}{4}\vartheta^{M\,A}\vartheta_M{}^B = 0\,.$$

Now we can repeat the procedure of the electric case:

First we construct derivatives $\mathfrak{D}$

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covariant under

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which only works if $\vartheta_M{}^A$ is an invariant tensor

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Before moving forward, we must impose another constraint on the embedding tensor on top of the two quadratic ones $Q_{MN}{}^A = Q^{AB} = 0$:

$$L_{MNP} \equiv X_{(MNP)} = \vartheta_{(M}{}^A T_{A\,NP)} = 0 \,.$$

This *linear* or *representation constraint* is based on supergravity and eliminates certain possible representations of the embedding tensor . On the other hand, we cannot construct gauge -covariant 2-form field strengths F^M without it!

4 – The 4-d tensor hierarchy

To construct the gauge -covariant 2-form field strengths F^M we take the covariant derivative of the gauge -covariant "field strength" $\mathcal{D}Z^i$:

$$\mathfrak{D}\mathfrak{D}Z^i = [dA^M + \tfrac{1}{2}X_{NP}{}^M A^N \wedge A^P]\vartheta_M{}^A k_A{}^i\,,$$

which suggests the definition

$$F^M \equiv dA^M + \tfrac{1}{2}X_{NP}{}^M A^N \wedge A^P + \Delta F^M\,, \qquad \vartheta_M{}^A \Delta F^M = 0\,,$$

so we have the Bianchi identity

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$\delta_{\Lambda}B_{A}$ is determined by the gauge -covariance of F^{M} <u>plus</u> $\delta B_{A} \sim d\Lambda_{A}$.

4 – The 4-d tensor hierarchy

To construct the gauge -covariant 2-form field strengths F^M we take the covariant derivative of the gauge -covariant "field strength" $\mathcal{D}Z^i$:

$$\mathfrak{D}\mathfrak{D}Z^i = [dA^M + \tfrac{1}{2}X_{NP}{}^M A^N \wedge A^P]\vartheta_M{}^A k_A{}^i ,$$

which suggests the definition

$$F^M \equiv dA^M + \tfrac{1}{2}X_{NP}{}^M A^N \wedge A^P + \Delta F^M , \qquad \vartheta_M{}^A \Delta F^M = 0 ,$$

so we have the Bianchi identity

$$\boxed{\mathfrak{D}\mathfrak{D}Z^i = F^M \vartheta_M{}^A k_A{}^i \ .}$$

Using the constraint $Q^{AB} \equiv \frac{1}{4}\vartheta^{MA}\vartheta_M{}^B = 0$ the natural solution is

$$\Delta F^M = -\tfrac{1}{2}\vartheta^{MA} B_A \equiv Z^{MA} B_A .$$

$\delta_\Lambda B_A$ is determined by the gauge -covariance of F^M plus $\delta B_A \sim d\Lambda_A$.

But we do not need it to move forward.

If we take the covariant derivative of the gauge -covariant 2-form field strength F^{M} we find

$$\mathfrak{D}F^{M} = Z^{MA}\{\mathfrak{D}B_{A} + T_{A\,RS}A^{R} \wedge [dA^{S} + \tfrac{1}{3}X_{NP}{}^{S}A^{N} \wedge A^{P}]\}\,.$$

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The gauge -covariance of the l.h.s. suggests the definition

$$H_A = \mathfrak{D}B_A + T_{A\,RS}A^R \wedge [dA^S + \tfrac{1}{3}X_{NP}{}^S A^N \wedge A^P] + \Delta H_A\,, \quad \text{where} \quad Z^{MA}\Delta H_A = 0\,.$$

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Using the constraint

$$Q_{MN}{}^{A} = \vartheta_{M}{}^{B}(T_{B\,N}{}^{P}\vartheta_{P}{}^{A} - \vartheta_{N}{}^{C}f_{BC}{}^{A}) \equiv 2Z_{M}{}^{A}Y_{AN}{}^{P} = 0$$

the natural solution for $Z^{MA}\Delta H_{A} = Z^{MA}\Delta B_{A} = 0$ is

$$\Delta H_{A} \equiv Y_{AM}{}^{C}C_{C}{}^{M}\,.$$

$\delta_{\Lambda}C_{C}{}^{M}$ is fully determined by the gauge -covariance of H_{A}.

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If we take the covariant derivative of the gauge -covariant 3-form field strength H_A we find

$$\mathfrak{D}H_A - T_{A\,MN}F^M \wedge F^N = Y_{AM}{}^C\{\mathfrak{D}C_C{}^M + F^M \wedge B_C + \cdots\}\,.$$

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$$G_C{}^M = \mathfrak{D}C_C{}^M + F^M \wedge B_C + \cdots + \Delta G_C{}^M\,, \qquad \text{where} \qquad Y_{AM}{}^C \Delta G_C{}^M = 0\,.$$

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To determine $\Delta G_C{}^M$ we need to find invariant tensors that vanish upon contraction with $Y_{AM}{}^C$. They appear automatically when we take the gauge -covariant derivative of the Bianchi identity and $G_C{}^M$ (if we "forget" we are in 4 dimensions!).

Acting with $\mathfrak{D}$ on the Bianchi identity of H_A we find

$$Y_{AM}{}^{C}\{\mathfrak{D}G_{C}{}^{M} - F^{M} \wedge H_{A}\} = 0\,, \;\; \Rightarrow \;\; \mathfrak{D}G_{C}{}^{M} = F^{M} \wedge H_{A} + \Delta\mathfrak{D}G_{C}{}^{M}\,,$$

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Acting with $\mathfrak{D}$ on the above identity we find

$$\mathfrak{D}\Delta\mathfrak{D}G_{C}{}^{M} = W_{C}{}^{MAB}H_A\wedge H_B + W_{CNPQ}{}^{M}F^{N}\wedge F^{P}\wedge F^{Q} + W_{CNP}{}^{EM}F^{N}\wedge G_{E}{}^{P}\,.$$

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This implies that there are 3 such tensors $W_{C}{}^{MAB}, W_{CNPQ}{}^{M}, W_{CNP}{}^{EM}$ that vanish contracted with $Y_{AM}{}^{C}$ and which we can use to build $\Delta G_{C}{}^{M}$.

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The natural solution is

$$\Delta G_{C}{}^{M} = W_{C}{}^{MAB}D_{AB} + W_{CNPQ}{}^{M}D^{NPQ} + W_{CNP}{}^{EM}D_{E}{}^{NP}\,,$$

and $\delta_{\Lambda}D_{AB}, \delta_{\Lambda}D^{NPQ}, \delta_{\Lambda}D_{E}{}^{NP}$ will follow from the gauge -covariance of $G_{C}{}^{M}$.

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\begin{aligned}
\delta_\Lambda A^M &= -\mathfrak{D}\Lambda^M - Z^{MA}\Lambda_A\,,\\
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- The covariant $(p+2)$-form field strengths $F^M, H_A, G_C{}^M$, related by Bianchi identities.

This system is known as the (4-dimensional) *tensor hierarchy*.

It is *universal*: it exists for all 4-dimensional FTs with gauge symmetry.

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But, what does it mean?
What is the meaning of the additional fields?

5 – The meaning of the $d = 4$ tensor hierarchy

These are the fields that we need to make a general gauging of any FT.

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☞ These two duality relations together with the Bianchi identity $\mathfrak{D}F^M = Z^{MA}H_A$ give a set of electric -magnetic duality -covariant Maxwell equations:

$$\mathfrak{D}F^\Lambda = -\tfrac{1}{4}\vartheta_\Lambda{}^A \star j_A \quad , \qquad \mathfrak{D}G_\Lambda = \tfrac{1}{4}\vartheta^{\Lambda\, A} \star j_A \quad .$$

☞ The 3-forms $C_C{}^M$ must be dual to constants: the embedding tensor $\vartheta_M{}^C$. This duality is expressed through the formula

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This equation is similar to the consistency condition (gauge or Noether identity) that Noether currents must satisfy off-shell in FTs with gauge invariance:

$$\mathfrak{D} \star j_A = -2(k_A{}^i \mathcal{E}_i + \text{c.c.}) + 4T_{A\,MN} G^M \wedge G^N + \star Y_A{}^{MC} \frac{\partial V}{\partial \vartheta_M{}^C} \, ,$$

where $\mathcal{E}_i$ is the e.o.m. of Z^i. Both equations, together, imply

$$k_A{}^i \mathcal{E}_i + \text{c.c.} = 0 \quad ,$$

which is equivalent to the scalar e.o.m. for symmetric σ-models.

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To confirm this interpretation we must construct a gauge -invariant action for **all** these fields, including the embedding tensor .

This gauge -invariant action is given by

$$
\begin{aligned}
S[g_{\mu\nu}, Z^i, A^M, B_A, C_A{}^M, D_E{}^{NP}, D_{AB}, D^{MNP}, \vartheta_M{}^A] \quad = \\
\int \{ -2\mathcal{G}_{ij^*}\mathfrak{D}Z^i \wedge \star\mathfrak{D}Z^{*\,j^*} + 2F^\Sigma \wedge G_\Sigma - \star V \\
-4Z^{\Sigma A}B_A \wedge \left(F_\Sigma - \tfrac{1}{2}Z_\Sigma{}^B B_B\right) - \tfrac{4}{3}X_{[MN]\Sigma}A^M \wedge A^N \wedge \left(F^\Sigma - Z^{\Sigma B}B_B\right) \\
-\tfrac{2}{3}X_{[MN]}{}^\Sigma A^M \wedge A^N \wedge \left(dA_\Sigma - \tfrac{1}{4}X_{[PQ]\Sigma}A^P \wedge A^Q\right) \\
-2\mathfrak{D}\vartheta_M{}^A \wedge (C_A{}^M + A^M \wedge B_A) \\
+2Q_{NP}{}^E(D_E{}^{NP} - \tfrac{1}{2}A^N \wedge A^P \wedge B_E) + 2Q^{AB}D_{AB} + 2L_{MNP}D^{MNP} \} \,.
\end{aligned}
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We are going to review ungauged $N = 1$ supergravity and its global symmetries and then we are going to gauge them using the embedding tensor formalism.

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All fermions are represented by chiral 4-component spinors:

$$\gamma_5 \psi_{\mu} = -\psi_{\mu}\,, \text{etc.}$$

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The spinors transform as *sections* of the bundle: under Kähler transformations

$$\delta_\lambda \mathcal{K} = \lambda(Z) + \lambda^*(Z^*)\,, \qquad \delta_\lambda \psi_\mu = -\tfrac{1}{4}[\lambda(Z) - \lambda^*(Z^*)]\psi_\mu\,,$$

and their covariant derivatives contain the pullback of the Kähler connection 1-form $\mathcal{Q} \equiv \mathcal{Q}_i dZ^i + \mathcal{Q}_{i^*} dZ^{*i^*}$ e.g.

$$\mathcal{D}_\mu \psi_\nu = \{\nabla_\mu + \tfrac{i}{2}\mathcal{Q}_\mu\}\psi_\nu\,.$$

$N = 1$ supergravity allows for an arbitrary holomorphic kinetic matrix $f_{\Lambda\Sigma}(Z)$ for the vector fields which occurs in the action in the terms

$$-2\Im\mathrm{m} f_{\Lambda\Sigma} F^{\Lambda} \wedge \star F^{\Sigma} + 2\Re\mathrm{e} f_{\Lambda\Sigma} F^{\Lambda} \wedge F^{\Sigma}\,, \qquad\qquad F^{\Lambda} \equiv dA^{\Lambda}\,.$$

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Finally, ungauged $N=1$ supergravity allows for a holomorphic superpotential $\mathcal{W}(Z)$ which appears through the covariantly holomorphic section of Kähler weight $(1,-1)$ $\mathcal{L}(Z,Z^*)$:

$$\mathcal{L}(Z,Z^*) = \mathcal{W}(Z) e^{\mathcal{K}/2} \,, \qquad \mathcal{D}_{i^*} \mathcal{L} = 0 \,,$$

which couples to the fermions in various ways and gives rise to the scalar potential

$$V_{\text{u}}(Z,Z^*) = -24|\mathcal{L}|^2 + 8\mathcal{G}^{ij^*} \mathcal{D}_i \mathcal{L} \mathcal{D}_{j^*} \mathcal{L}^* \,.$$

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$$\begin{aligned} S_{\mathrm{u}}[g_{\mu\nu}, Z^i, A^{\Lambda}] \;\; = \;\; & \int \{\star R - 2\mathcal{G}_{ij^*} dZ^i \wedge \star dZ^{*\,j^*} - 2\Im\mathrm{m} f_{\Lambda\Sigma} F^{\Lambda} \wedge \star F^{\Sigma} \\ & + 2\Re\mathrm{e} f_{\Lambda\Sigma} F^{\Lambda} \wedge F^{\Sigma} - \star V_{\mathrm{u}}(Z,Z^*)\} \,. \end{aligned}$$

To first order in fermions , the supersymmetry transformations for the fermions are

$$\delta_{\epsilon}\psi_{\mu} = \mathcal{D}_{\mu}\epsilon + i\mathcal{L}\gamma_{\mu}\epsilon^{*} = \left[\nabla_{\mu} + \tfrac{i}{2}\mathcal{Q}_{\mu}\right]\epsilon + i\mathcal{L}\gamma_{\mu}\epsilon^{*}\,,$$

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This is not the full FT but this information (bosonic action and supersymmetry transformations) is enough to reconstruct it.

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- ☞ The superpotential $\mathcal{L}(Z, Z^*)$ is not a fundamental field and this phase change is not a symmetry unless it can be reabsorbed into a transformation of the scalars.
- ☞ But this would mean that we are dealing with a $A = \mathrm{a}$ symmetry and we can say that a non-vanishing superpotential breaks $U(1)_R$ and we cannot gauge it.

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- Then, the spinors ’ covariant derivatives take the form

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$$\mathcal{L} \neq 0\,, \;\; \Rightarrow \vartheta_M{}^A(\delta_A{}^{\underline{a}}\mathcal{P}_{\underline{a}} + \delta_A{}^{\#}\mathcal{P}_{\#}) = 0\;\;.$$

9 – The $N = 1, d = 4$ bosonic tensor hierarchy

We have found that, for non-vanishing superpotential , the embedding tensor must satisfy another constraint

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This will happen in $N = 1$ supergravity if we find new Stückelberg shifts

$$\boxed{\delta' B_A \sim \delta_h B_A + Y_A \Lambda \quad \text{and} \quad \delta' C_C{}^M = \delta_h C_C{}^M + Y_C \Lambda^M\;.}$$

10 – The $N = 1, d = 4$ supersymmetric tensor hierarchy

As a first step to include the tensor hierarchy fields into $N = 1$ supergravity we are going to construct supersymmetry transformation rules such that the local supersymmetry algebra, to leading order in fermions , closes on the new fields up to duality relations.

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As a first step to include the tensor hierarchy fields into $N = 1$ supergravity we are going to construct supersymmetry transformation rules such that the local supersymmetry algebra, to leading order in fermions , closes on the new fields up to duality relations.

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Observe that we are going to obtain, independently, the gauge transformations of the fields and will confirm or refute the hierarchy's results.

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$$\delta_\eta \chi^i = i \not{\mathfrak{D}} Z^i \eta^* + 2\mathcal{G}^{ij^*}\mathcal{D}_{j^*}\mathcal{L}^*\eta\,, \qquad \mathfrak{D} Z^i = dZ^i + A^M \vartheta_M{}^A k_A{}^i\,.$$

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We find the expected result

$$\begin{aligned}
[\delta_\eta\,,\,\delta_\epsilon] Z^i &= \delta_{\text{g.c.t.}} Z^i + \delta_h Z^i\,,\\
\delta_{\text{g.c.t.}} Z^i &= \pounds_\xi Z^i = +\xi^\mu \partial_\mu Z^i\,,\\
\delta_h Z^i &= \Lambda^M \vartheta_M{}^A k_A{}^i\,,\\
\xi^\mu &\equiv \tfrac{i}{4}(\bar{\epsilon}\gamma^\mu\eta^* - \bar{\eta}\gamma^\mu\epsilon^*)\,,\\
\Lambda^M &\equiv \xi^\mu A^M{}_\mu\,.
\end{aligned}$$

The 1-forms A^M

We introduce supersymmetric partners λ_Σ for the magnetic 1-forms A_Σ and make the symplectic -covariant Ansatz

$$\begin{aligned}\delta_\epsilon A^M{}_\mu &= -\tfrac{i}{8}\bar{\epsilon}^*\gamma_\mu\lambda^M + \text{c.c.}\,,\\ \delta_\epsilon\lambda^M &= \tfrac{1}{2}\left[\not{F}^{M+} + i\mathcal{D}^M\right]\epsilon\,,\end{aligned}$$

where we have defined the symplectic vector

$$\mathcal{D}^M \equiv \begin{pmatrix}\mathcal{D}^\Lambda\\ \mathcal{D}_\Lambda\end{pmatrix} \equiv \begin{pmatrix}\mathcal{D}_\Lambda\\ f_{\Lambda\Sigma}\mathcal{D}^\Sigma\end{pmatrix}\,, \qquad \mathcal{D}^\Lambda = -\Im\text{m} f^{\Lambda\Sigma}\,(\vartheta_\Sigma{}^A + f^*_{\Sigma\Omega}\vartheta^{\Omega\,A})\mathcal{P}_A\,.$$

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where

$$\Lambda_A \equiv -T_{A\,MN}A^N\Lambda^M + b_A - \mathcal{P}_A\xi\,, \qquad b_{A\,\mu} \equiv B_{A\,\mu\nu}\xi^\nu\,.$$

The 2-forms B_A

We introduce the supersymmetric partners ζ_A, φ_A (linear supermultiplets)

$$\begin{aligned}
\delta_\epsilon \zeta_A &= -i[\tfrac{1}{12}\not{H}'_A + \not{\mathfrak{D}}\varphi_A]\epsilon^* - 4\delta_A{}^{\mathbf{a}}\varphi_{\mathbf{a}}\mathcal{L}^*\epsilon\,, \\
\delta_\epsilon B_{A\,\mu\nu} &= \tfrac{1}{4}[\bar{\epsilon}\gamma_{\mu\nu}\zeta_A + \text{c.c.}] - i[\varphi_A\bar{\epsilon}^*\gamma_{[\mu}\psi_{\nu]} - \text{c.c.}] + 2T_{A\,MN}A^M{}_{[\mu}\delta_\epsilon A^N{}_{\nu]}\,, \\
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but, again, we do not need them to show that

$$[\delta_\eta\,,\,\delta_\epsilon]B_A = \delta_{\text{g.c.t.}}B_A + \delta'_h B_A\,,$$

which **proofs** the existence of an extra Stückelberg shift in B_A.

The 3-forms $C_A{}^M$

In this case we do not introduce any supersymmetric partners. We just make the Ansatz

$$\delta_\epsilon C_A{}^M{}_{\mu\nu\rho} = -\tfrac{i}{8}[\mathcal{P}_A \bar{\epsilon}^* \gamma_{\mu\nu\rho}\lambda^M - \text{c.c.}] - 3B_{A\,[\mu\nu|}\delta_\epsilon A^M{}_{|\rho]} - 2T_{A\,PQ}A^M{}_{[\mu}A^P{}_{\nu|}\delta_\epsilon A^Q{}_{|\rho]}\,.$$

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This corresponds to a scalar potential of the form

$$V_{\mathrm{e-mg}} = V_{\mathrm{u}} - \tfrac{1}{2}\Re\mathrm{e}\,\mathcal{D}^M\vartheta_M{}^A\mathcal{P}_A = V_{\mathrm{u}} + \tfrac{1}{2}\mathcal{M}^{MN}\vartheta_M{}^A\vartheta_N{}^B\mathcal{P}_A\mathcal{P}_B\,,$$

where

$$(\mathcal{M}^{MN}) \equiv \begin{pmatrix} I^{\Lambda\Sigma} & I^{\Lambda\Omega}R_{\Omega\Sigma} \\ R_{\Lambda\Omega}I^{\Omega\Sigma} & I_{\Lambda\Sigma} + R_{\Lambda\Omega}I^{\Omega\Gamma}R_{\Gamma\Sigma} \end{pmatrix}\,, \qquad \begin{aligned} f_{\Lambda\Sigma} &\equiv R_{\Lambda\Sigma} + iI_{\Lambda\Sigma}\,, \\ I^{\Lambda\Omega}I_{\Omega\Sigma} &\equiv \delta^\Lambda{}_\Sigma\,, \end{aligned}$$

so it is manifestly symplectic -invariant, as it must.

The 3-forms C, C'

The consistency of the previous results requires the existence of a 3-form C transforming under the extra Stückelberg shift of B_A.

$$\delta_\epsilon C_{\mu\nu\rho} = -3i\mathbf{g}\mathcal{L}\,\bar{\epsilon}^{*}\gamma_{[\mu\nu}\psi^{*}{}_{\rho]} - \tfrac{1}{2}\mathbf{g}\mathcal{D}_i\mathcal{L}\bar{\epsilon}^{*}\gamma_{\mu\nu\rho}\chi^{i} + \text{c.c.}\,,$$

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$$G' = \star\mathbf{g}(-24|\mathcal{L}|^2 + 8\mathcal{G}^{ij^*}\mathcal{D}_i\mathcal{L}\mathcal{D}_{j^*}\mathcal{L}^*)\,.$$

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   $$G' = \star\mathbf{g}\left(-24|\mathcal{L}|^2 + 8\mathcal{G}^{ij^*}\mathcal{D}_i\mathcal{L}\mathcal{D}_{j^*}\mathcal{L}^*\right).$$

   If we rescale the superpotential by $\mathcal{L} \to \mathbf{g}\mathcal{L}$, the above duality relation takes the standard form

   $$G' = \tfrac{1}{2}\star\frac{\partial V_{\text{e}-\text{mg}}}{\partial\mathbf{g}}\,,$$

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So, what is the 3-form C dual to?

The 4-forms $D_{AB}, D^{NPQ}, D_E{}^{NP}, D^M$

The calculations become horribly complicated and we only check the closure of the local supersymmetry algebra in the ungauged $\vartheta_M{}^A = 0$ case when there are no symmetries acting on the 1-forms i.e. $T_{A\,M}{}^N = 0$.

The supersymmetry transformations are

$$\begin{aligned}
\delta_\epsilon D_{AB} &= -\tfrac{i}{2} \star \mathcal{P}_{[A}\partial_i \mathcal{P}_{B]}\bar{\epsilon}\chi^i + \text{c.c.} - B_{[A} \wedge \delta_\epsilon B_{B]} \,, \\
\delta_\epsilon D^{NPQ} &= 10 A^{(N} \wedge F^P \wedge \delta_\epsilon A^{Q)} \,, \\
\delta_\epsilon D_E{}^{NP} &= C_E{}^P \wedge \delta_\epsilon A^N \,. \\
\delta_\epsilon D^M &= -\tfrac{i}{2} \star \mathcal{L}^* \bar{\epsilon}\lambda^M + \text{c.c.} + C \wedge \delta_\epsilon A^M \,.
\end{aligned}$$

This proves that D^M can be consistently added to the supersymmetric theory. Its role in the action will be that of Lagrange multiplier of the constraint Q_M.

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pp-waves $(e^a{}_\mu)$, strings (B_A) and domain walls (C, C').

We are going to focus on the domain walls associated to the 3-form C' since we need to know the associated deformation parameter in order to couple C to supergravity . We consider the ungauged theory with only chiral supermultiplets and superpotential

12 – Domain-wall solutions of $N = 1$ supergravity

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The metric of a 4-d domain-wall solution can always be written in the form

$$ds^2 = V\eta_{\mu\nu}dx^\mu dx^\nu = V(y)[\eta_{mn}dx^m dx^n - dy^2]\,, \qquad m,n = 0,1,2\,.$$

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If the $Z^i = Z^i(y)$ the gravitino Killing spinor equation $\delta_\epsilon\psi_\mu = 0$ is be solved by

$$(e^{-i\alpha/2}\epsilon) \pm i\gamma^{012}(e^{-i\alpha/2}\epsilon)^* = 0\,, \qquad e^{i\alpha} \equiv \mathcal{L}/|\mathcal{L}|\,.$$

and $V(y)$ satisfies the "*V flow equation*"

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These two first-order *flow equations* imply the second-order supergravity equations of motion.

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suggest that the kinetic term contains a Z-dependent factor (*tension*) which is a real function of $\mathcal{L}$

$$S_{\rm DW} = \int d^3\xi \left\{|\mathcal{L}|\sqrt{|g_{(3)}|} + \frac{1}{3!}\beta\epsilon^{mnp}C'_{mnp}\right\}\,,$$

where $\beta \in \mathbb{R}$, $|g_3|$ the determinant of the pullback $g_{(3)\,mn}$ of the spacetime metric over the 3-dimensional worldvolume and C'_{mnp} is the pullback of the 3-form.

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suggest that the kinetic term contains a Z-dependent factor (*tension*) which is a real function of $\mathcal{L}$

$$S_{\rm DW} = \int d^3\xi \left\{|\mathcal{L}|\sqrt{|g_{(3)}|} + \frac{1}{3!}\beta\epsilon^{mnp}C'_{mnp}\right\}\,,$$

where $\beta \in \mathbb{R}$, $|g_3|$ the determinant of the pullback $g_{(3)\,mn}$ of the spacetime metric over the 3-dimensional worldvolume and C'_{mnp} is the pullback of the 3-form.

In the static gauge $\partial X^\mu/\partial\xi^m = \delta^\mu{}_m$ it can be seen that this action is invariant to lowest order in fermions under the supersymmetry transformations of $g_{\mu\nu}, Z^i, C'_{\mu\nu\rho}$ if $\beta = \pm 1/4$ and the spinors satisfy the BPS domain-wall projection $(e^{-i\alpha/2}\epsilon) \pm i\gamma^{012}(e^{-i\alpha/2}\epsilon)^* = 0$.

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Thus, we consider the bulk supergravity action,

$$S_{\text{bulk}} = \frac{1}{\kappa^2}\int d^4x\sqrt{|g|}\left[R + 2\mathcal{G}_{ij^*}\partial_\mu Z^i \partial^\mu Z^{*\,j^*} - \mathbf{g}^2(x)V(Z,Z^*) - \tfrac{1}{3\sqrt{|g|}}\epsilon^{\mu\nu\rho\sigma}\partial_\mu \mathbf{g}(x) C'_{\nu\rho\sigma}\right]$$

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and the brane source action

$$S_{\text{brane}} = \int d^4x \mathbf{f}(y)\left\{|\mathcal{L}|\sqrt{|g_{(3)}|} \pm \frac{1}{4!}\epsilon^{mnp}C'_{\underline{mnp}}\right\},$$

where $\mathbf{f}(y)$ is a distribution function of domain walls common transverse direction $x^3 \equiv y$: $\mathbf{f}(y) = \delta^{(1)}(y - y_0)$ for a single domain wall placed at $y = y_0$ etc.

The equations of motion that follow from $S \equiv S_{\text{bulk}} + S_{\text{brane}}$ are

$$\begin{aligned}
\mathcal{E}_{\mathbf{g}}^{\mu\nu} &= \tfrac{\kappa^2}{2}\mathbf{f}(y)|\mathcal{L}|\frac{\sqrt{|g_{(3)}|}}{\sqrt{|g|}}g_{(3)}^{mn}\delta_m{}^{\mu}\delta_n{}^{\nu}\,, \\
\mathcal{G}^{ij^*}\mathcal{E}_{\mathbf{g}\,i^*} &= \tfrac{\kappa^2}{8}\mathbf{f}(y)\frac{\sqrt{|g_{(3)}|}}{\sqrt{|g|}}e^{i\alpha}\mathcal{N}^i\,, \\
\epsilon^{\mu\nu\rho\sigma}\partial_\sigma\mathbf{g}(x) &= \mp\tfrac{\kappa^2}{8}\mathbf{f}(y)\epsilon^{mnp}\delta_m{}^{\mu}\delta_n{}^{\nu}\delta_p{}^{\rho}\,, \\
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The fourth equation is that of $\mathbf{g}(x)$ and y states that C' is the dual of the scalar potential.

It can now be checked that the Einstein and scalar equations of motion are identically satisfied if $V(y)$ and the scalars $Z^i(y)$ satisfy the *sourceful flow equations*

$$\partial_{\underline{y}} Z^i = \pm \mathbf{g}(y) e^{i\alpha} \mathcal{N}^i V^{1/2} ,$$

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These equations can be derived from the modified fermion supersymmetry transformations

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A fully supersymmetric "democratic" formulation of $N = 1$ $d = 4$ supergravity including all higher-rank forms and local coupling constants is necessary to accomodate these modifications.

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- Since all the null-class supersymmetric solutions of 4-d supergravities can be related to an $N = 1$ truncation, we can use these results to construct domain-wall solutions and effective actions for $N > 1$.