4-Dimensional Gauge Theories and Tensor Hierarchies

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Talk given on the February 3rd 2009 at the *5th (1st Iberian) Workshop on Gravitational Aspects of Strings and Branes*, Gijón, Spain

Based on 0901.2054 and work in preparation.

Work done in collaboration with *E. Bergshoeff, O. Hohm* (U. Groningen) *J. Hartong* (U. Bern) and *M. Hübscher* (IFT UAM/CSIC, Madrid)

Plan of the Talk:

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We are going to use the embedding tensor method to find all the $(p+1)$-form potentials and the corresponding democratic formulations of 4-dimensional supergravities (or any other 4-dimensional field theory with gauge symmetry).

The next steps in this program will be:

[a] So far, only maximal and half-maximal supergravities have been studied from this point of view de Wit, Samtleben & Trigiante, arXiv:hep-th/0412173, Samtleben & Weidner arXiv:hep-th/0506237, Schon & Weidner, arXiv:hep-th/0602024, de Wit, Samtleben & Trigiante, arXiv:0705.2101, Bergshoeff, Gomis, Nutma & Roest, arXiv:0711.2035, de Wit, Nicolai & Samtleben, arXiv:0801.1294.

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Then we will extend the formalism to electric and magnetic gaugings of general (perturbative and non-perturbative) symmetries. We will find the need to introduce higher-rank form potentials defining a structure called *tensor hierarchy* de Wit & Samtleben, arXiv:hep-th/0501243, de Wit, Samtleben & Trigiante, arXiv:hep-th/0507289.

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Then, we are going to find all the fields of the tensor hierarchy for arbitrary 4-dimensional field theories and we are going to construct a gauge -invariant action for all those fields[a].

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2 – The embedding tensor method: electric gaugings

Consider a general ($N = 1$ supergravity -inspired) 4-dimensional ungauged theory with bosonic fields $\{Z^i, A^\Lambda\}$ (the metric plays no relevant role here)

$$S_\mathrm{u}[Z^i, A^\Lambda] = \int \{-2\mathcal{G}_{ij^*} dZ^i \wedge \star dZ^{*\,j^*} - 2\Im\mathrm{m} f_{\Lambda\Sigma} F^\Lambda \wedge \star F^\Sigma + 2\Re\mathrm{e} f_{\Lambda\Sigma} F^\Lambda \wedge F^\Sigma - \star V_\mathrm{u}(Z, Z^*)\}\,.$$

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Let us assume this action is invariant under the global transformations

$$\begin{aligned}
\delta_\alpha Z^i &= \alpha^A k_A{}^i(Z)\,,\\
\delta_\alpha f_{\Lambda\Sigma} &\equiv -\alpha^A \pounds_A f_{\Lambda\Sigma} = \alpha^A [T_{A\,\Lambda\Sigma} - 2T_{A\,(\Lambda}{}^\Omega f_{\Sigma)\Omega}]\,,\\
\delta_\alpha A^\Lambda &= \alpha^A T_{A\,\Sigma}{}^\Lambda A^\Sigma\,.
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Each embedding tensor $\vartheta_\Lambda{}^A$ defines a possible identification:

$$\alpha^A(x) \equiv \Lambda^\Sigma \vartheta_\Sigma{}^A \,, \qquad\qquad A^A \equiv A^\Sigma \vartheta_\Sigma{}^A \,.$$

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Now we construct derivatives $\mathfrak{D}$

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covariant under

$$\begin{aligned} \delta_\Lambda Z^i &= \Lambda^\Sigma \vartheta_\Sigma{}^A k_A{}^i(Z) \,, \\ \delta_\Lambda A^\Sigma &= -\mathfrak{D}\Lambda^\Sigma \equiv -(d\Lambda^\Sigma + \vartheta_\Lambda{}^A T_{A\,\Omega}{}^\Sigma A^\Lambda \Lambda^\Omega) \,. \end{aligned}$$

This only works if $\vartheta_\Lambda{}^A$ is an invariant tensor

$$\delta_\Lambda \vartheta_\Sigma{}^A = -\Lambda^\Omega Q_{\Omega\Sigma}{}^A = 0\,, \qquad Q_{\Sigma\Lambda}{}^A \equiv \vartheta_\Sigma{}^B T_{B\,\Lambda}{}^\Omega \vartheta_\Omega{}^A - \vartheta_\Sigma{}^B \vartheta_\Lambda{}^C f_{BC}{}^A\,.$$

$Q_{\Omega\Sigma}{}^A = 0$ is known as the *quadratic constraint* in the embedding tensor formalism.

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$$X_{\Sigma\Lambda}{}^{\Omega} \equiv \vartheta_{\Sigma}{}^{B}T_{B\,\Lambda}{}^{\Omega}\,,$$

which satisfy the algebra

$$[T_{A}, T_{B}] = -f_{AB}{}^{C}\,, \;\Rightarrow\; [X_{\Sigma}, X_{\Lambda}] = -X_{\Sigma\Lambda}{}^{\Omega}X_{\Omega}\,,$$

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Then we construct the covariant 2-form field strengths

$$F^\Lambda = dA^\Lambda + \tfrac{1}{2} {X_{\Sigma\Omega}}^\Lambda A^\Sigma \wedge A^\Omega\,,$$

and the gauge -invariant action of the electrically gauged theory takes the form

$$S_{\rm eg}[Z^i, A^\Lambda] = \int \{-2\mathcal{G}_{ij^*} \mathfrak{D} Z^i \wedge \star \mathfrak{D} Z^{*\,j^*} - 2\Im\mathrm{m} f_{\Lambda\Sigma} F^\Lambda \wedge \star F^\Sigma + 2\Re\mathrm{e} f_{\Lambda\Sigma} F^\Lambda \wedge F^\Sigma - \star V_{\rm eg}(Z, Z^*)\}$$

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- The theory (equations of motion) has more non-perturbative global symmetries that can be gauged . They include electric -magnetic duality rotations:

$$\begin{aligned}
\delta_\alpha Z^i &= \alpha^A k_A{}^i(Z)\,,\\
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\delta_\alpha \begin{pmatrix} A^\Lambda \\ A_\Lambda \end{pmatrix} &= \alpha^A \begin{pmatrix} T_{A\,\Sigma}{}^\Lambda & T_A{}^{\Sigma\Lambda} \\ T_{A\,\Sigma\Lambda} & T_A{}^\Sigma{}_\Lambda \end{pmatrix} \begin{pmatrix} A^\Sigma \\ A_\Sigma \end{pmatrix}.
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Now we need to relate the α^{A} to the gauge parameters of the 1-forms Λ^{Λ} or Λ_{Λ} We need new (magnetic) components for the embedding tensor : $\vartheta^{\Lambda\, A}$. Then

$$\alpha^{A}(x) \equiv \Lambda^{\Sigma}\vartheta_{\Sigma}{}^{A} + \Lambda_{\Sigma}\vartheta^{\Sigma\, A}\,, \qquad A^{A} \equiv A^{\Sigma}\vartheta_{\Sigma}{}^{A} + A_{\Sigma}\vartheta^{\Sigma\, A}\,.$$

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Knowing (Gaillard & Zumino) that the T_A matrices either belong to $\mathfrak{sp}(2n_V,\mathbb{R})$ or vanish, we introduce the symplectic notation

$$A^M \equiv \begin{pmatrix} A^\Sigma \\ A_\Sigma \end{pmatrix} \qquad \vartheta_M{}^A \equiv (\vartheta_\Sigma{}^A, \vartheta^{\Sigma\,A}) \qquad \alpha^A(x) \equiv \Lambda^M \vartheta_M{}^A\,,$$

$$(T_{A\,M}{}^N) \equiv \begin{pmatrix} T_{A\,\Sigma}{}^\Lambda & T_A{}^{\Sigma\Lambda} \\ T_{A\,\Sigma\Lambda} & T_A{}^\Sigma{}_\Lambda \end{pmatrix}.$$

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We cannot gauge simultaneously a 1-form and its dual de Wit, Samtleben & Trigiante, arXiv:`hep-th/0507289`:

$$Q^{AB} \equiv \tfrac{1}{4}\vartheta^{M\,A}\vartheta_{M}{}^{B} = 0\,.$$

Now we can repeat the procedure of the electric case:

First we construct derivatives $\mathfrak{D}$

$$\mathfrak{D}Z^i \equiv dZ^i + A^M \vartheta_M{}^A k_A{}^i \,,$$

covariant under

$$\delta_\Lambda Z^i = \Lambda^M \vartheta_M{}^A k_A{}^i(Z) \,,$$

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Before moving forward, we must impose another constraint on the embedding tensor on top of the two quadratic ones $Q_{MN}{}^A = Q^{AB} = 0$:

$$L_{MNP} \equiv X_{(MNP)} = \vartheta_{(M}{}^A T_{A\,NP)} = 0 .$$

This *linear* or *representation constraint* is based on supergravity and eliminates certain possible representations of the embedding tensor . On the other hand, we cannot construct gauge -covariant 2-form field strengths F^M without it!

4 – The 4-d tensor hierarchy

To construct the gauge -covariant 2-form field strengths F^M we take the covariant derivative of the gauge -covariant "field strength" $\mathcal{D}Z^i$:

$$\mathcal{D}\mathcal{D}Z^i = [dA^M + \tfrac{1}{2}X_{NP}{}^M A^N \wedge A^P]\vartheta_M{}^A k_A{}^i \, ,$$

which suggests the definition

$$F^M \equiv dA^M + \tfrac{1}{2}X_{NP}{}^M A^N \wedge A^P + \Delta F^M \, , \qquad \vartheta_M{}^A \Delta F^M = 0 \, ,$$

so we have the Bianchi identity

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$\delta_\Lambda B_A$ is determined by the gauge -covariance of F^M plus $\delta B_A \sim d\Lambda_A$.

4 – The 4-d tensor hierarchy

To construct the gauge -covariant 2-form field strengths F^M we take the covariant derivative of the gauge -covariant "field strength" $\mathcal{D}Z^i$:

$$\mathcal{D}\mathcal{D}Z^i = [dA^M + \tfrac{1}{2}X_{NP}{}^M A^N \wedge A^P]\vartheta_M{}^A k_A{}^i\,,$$

which suggests the definition

$$F^M \equiv dA^M + \tfrac{1}{2}X_{NP}{}^M A^N \wedge A^P + \Delta F^M\,, \qquad \vartheta_M{}^A \Delta F^M = 0\,,$$

so we have the Bianchi identity

$$\boxed{\mathcal{D}\mathcal{D}Z^i = F^M \vartheta_M{}^A k_A{}^i\ .}$$

Using the constraint $Q^{AB} \equiv \frac{1}{4}\vartheta^{MA}\vartheta_M{}^B = 0$ the natural solution is

$$\Delta F^M = -\tfrac{1}{2}\vartheta^{MA}B_A \equiv Z^{MA}B_A\,.$$

$\delta_\Lambda B_A$ is determined by the gauge -covariance of F^M plus $\delta B_A \sim d\Lambda_A$.

But we do not need it to move forward.

If we take the covariant derivative of the gauge -covariant 2-form field strength F^M we find

$$\mathfrak{D}F^M = Z^{MA}\{\mathfrak{D}B_A + T_{A\,RS}A^R \wedge [dA^S + \tfrac{1}{3}X_{NP}{}^S A^N \wedge A^P]\}\,.$$

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The gauge -covariance of the l.h.s. suggests the definition

$$H_A = \mathfrak{D}B_A + T_{A\,RS}A^R \wedge [dA^S + \tfrac{1}{3}X_{NP}{}^S A^N \wedge A^P] + \Delta H_A\,, \qquad \text{where} \qquad Z^{MA}\Delta H_A = 0\,.$$

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Using the constraint

$$Q_{MN}{}^A = \vartheta_M{}^B(T_{B\,N}{}^P\vartheta_P{}^A - \vartheta_N{}^C f_{BC}{}^A) \equiv 2Z_M{}^A Y_{AN}{}^P = 0$$

the natural solution for $Z^{MA}\Delta H_A = Z^{MA}\Delta B_A = 0$ is

$$\Delta H_A \equiv Y_{AM}{}^C C_C{}^M\,.$$

$\delta_\Lambda C_C{}^M$ is fully determined by the gauge -covariance of H_A.

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If we take the covariant derivative of the gauge -covariant 3-form field strength H_A we find

$$\mathfrak{D}H_A - T_{A\,MN}F^M \wedge F^N = Y_{AM}{}^C\{\mathfrak{D}C_C{}^M + F^M \wedge B_C + \cdots\}\,.$$

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$$G_C{}^M = \mathfrak{D}C_C{}^M + F^M \wedge B_C + \cdots + \Delta G_C{}^M\,, \qquad \text{where} \qquad Y_{AM}{}^C \Delta G_C{}^M = 0\,.$$

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To determine $\Delta G_C{}^M$ we need to find invariant tensors that vanish upon contraction with $Y_{AM}{}^C$. They appear automatically when we take the gauge -covariant derivative of the Bianchi identity and $G_C{}^M$ (if we “forget” we are in 4 dimensions!).

Acting with $\mathfrak{D}$ on the Bianchi identity of H_A we find

$$Y_{AM}{}^{C}\{\mathfrak{D}G_C{}^{M} - F^{M} \wedge H_A\} = 0\,, \;\; \Rightarrow \;\; \mathfrak{D}G_C{}^{M} = F^{M} \wedge H_A + \Delta\mathfrak{D}G_C{}^{M}\,,$$

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Acting with $\mathfrak{D}$ on the above identity we find

$$\mathfrak{D}\Delta\mathfrak{D}G_C{}^{M} = W_C{}^{MAB}H_A \wedge H_B + W_{CNPQ}{}^{M}F^{N} \wedge F^{P} \wedge F^{Q} + W_{CNP}{}^{EM}F^{N} \wedge G_E{}^{P}\,.$$

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This implies that there are 3 such tensors $W_C{}^{MAB}, W_{CNPQ}{}^M, W_{CNP}{}^{EM}$ that vanish contracted with $Y_{AM}{}^C$ and which we can use to build $\Delta G_C{}^M$.

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This implies that there are 3 such tensors $W_C{}^{MAB}, W_{CNPQ}{}^{M}, W_{CNP}{}^{EM}$ that vanish contracted with $Y_{AM}{}^{C}$ and which we can use to build $\Delta G_C{}^{M}$.
The natural solution is

$$\Delta G_C{}^{M} = W_C{}^{MAB}D_{AB} + W_{CNPQ}{}^{M}D^{NPQ} + W_{CNP}{}^{EM}D_E{}^{NP}\,,$$

and $\delta_\Lambda D_{AB}, \delta_\Lambda D^{NPQ}, \delta_\Lambda D_E{}^{NP}$ will follow from the gauge -covariance of $G_C{}^{M}$.

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$$\begin{aligned}
\delta_\Lambda A^M &= -\mathfrak{D}\Lambda^M - Z^{MA}\Lambda_A\,,\\
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&\quad +\Lambda^N G_E{}^P\,,\\
\delta_\Lambda D^{NPQ} &= \mathfrak{D}\Lambda^{NPQ} - 3Z^{(N|A}\tilde{\Lambda}_A{}^{|PQ)} - 2A^{(N}\wedge dA^P\wedge\delta_\Lambda A^{Q)} - \tfrac{3}{4}X_{RS}{}^{(N}A^{P|}\wedge A^R\wedge A^S\wedge\delta_\Lambda A^{|Q)} - 3\Lambda^{(N}F^P
\end{aligned}$$

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This sytem is known as the (4-dimensional) *tensor hierarchy*.

It is *universal*: it exists for all 4-dimensional theories with gauge symmetry.

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But, what does it mean?
What is the meaning of the additional fields?

5 – The meaning of the $d = 4$ tensor hierarchy

These are the fields that we need to make a general gauging of any theory.

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- ☞ These two duality relations together with the Bianchi identity $\mathfrak{D}F^M = Z^{MA} H_A$ give a set of electric -magnetic duality -covariant Maxwell equations:

$$\mathfrak{D}F^\Lambda = -\tfrac{1}{4}\vartheta_\Lambda{}^A \star j_A \ , \qquad \mathfrak{D}G_\Lambda = \tfrac{1}{4}\vartheta^{\Lambda\,A} \star j_A \quad .$$

☞ The 3-forms $C_C{}^M$ must be dual to constants: the embedding tensor $\vartheta_M{}^C$. This duality is expressed through the formula

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$$\mathfrak{D} \star j_A = 4T_{A\,MN} G^M \wedge G^N + \star Y_A{}^{MC} \frac{\partial V}{\partial \vartheta_M{}^C} \quad .$$

  This equation is similar to the consistency condition (gauge or Noether identity) that Noether currents must satisfy off-shell in theories with gauge invariance:

$$\mathfrak{D} \star j_A = -2(k_A{}^i \mathcal{E}_i + \text{c.c.}) + 4T_{A\,MN} G^M \wedge G^N + \star Y_A{}^{MC} \frac{\partial V}{\partial \vartheta_M{}^C} \,,$$

  where $\mathcal{E}_i$ is the e.o.m. of Z^i. Both equations, together, imply

$$k_A{}^i \mathcal{E}_i + \text{c.c.} = 0 \quad ,$$

  which is equivalent to the scalar e.o.m. for symmetric σ-models.

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To show that this interpretation is right, we must construct a gauge -invariant action for these fields, including the embedding tensor .

The gauge -invariant action is

$$
\begin{aligned}
S \;=\; \int \Big\{ & -2\mathcal{G}_{ij^*}\mathfrak{D}Z^i \wedge \star\mathfrak{D}Z^{*\,j^*} + 2F^{\Sigma}\wedge G_{\Sigma} - \star V \\
& -4Z^{\Sigma A}B_A \wedge \left(F_{\Sigma} - \tfrac{1}{2}Z_{\Sigma}{}^{B}B_B\right) - \tfrac{4}{3}X_{[MN]\Sigma}A^M \wedge A^N \wedge \left(F^{\Sigma} - Z^{\Sigma B}B_B\right) \\
& -\tfrac{2}{3}X_{[MN]}{}^{\Sigma}A^M \wedge A^N \wedge \left(dA_{\Sigma} - \tfrac{1}{4}X_{[PQ]\Sigma}A^P \wedge A^Q\right) \\
& -2\mathfrak{D}\vartheta_M{}^A \wedge (C_A{}^M + A^M \wedge B_A) \\
& +2Q_{NP}{}^{E}(D_E{}^{NP} - \tfrac{1}{2}A^N \wedge A^P \wedge B_E) + 2Q^{AB}D_{AB} + 2L_{MNP}D^{MNP}\Big\}\,,
\end{aligned}
$$

☞ Finally, the indices of the 3 4-forms $D_{AB}, D^{NPQ}, D_E{}^{NP}$ are conjugate to those of the constraints $Q^{AB}, Q_{NPQ}, Q_{NP}{}^E$. They are Lagrange multipliers enforcing them.

To show that this interpretation is right, we must construct a gauge -invariant action for these fields, including the embedding tensor .

The gauge -invariant action is

$$
\begin{aligned}
S \;=\; \int \Big\{ & -2\mathcal{G}_{ij^*}\mathfrak{D}Z^i \wedge \star\mathfrak{D}Z^{*\,j^*} + 2F^\Sigma \wedge G_\Sigma - \star V \\
& -4Z^{\Sigma A}B_A \wedge \left(F_\Sigma - \tfrac{1}{2}Z_\Sigma{}^B B_B\right) - \tfrac{4}{3}X_{[MN]\Sigma}A^M \wedge A^N \wedge \left(F^\Sigma - Z^{\Sigma B}B_B\right) \\
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\end{aligned}
$$

And the e.o.m. in full glory are....

$$\tfrac{1}{2}\delta S/\delta Z^i = \mathcal{G}_{ij^*}\mathfrak{D}\star\mathfrak{D}Z^{*\,j^*} - \partial_i G_M{}^{+}\wedge G^{M+} - \star\tfrac{1}{2}\partial_i V\,,$$

$$-\tfrac{1}{4}\star\frac{\delta S}{\delta A^M} = \mathfrak{D}F_M - \tfrac{1}{4}\vartheta_M{}^A\star j_A - \tfrac{1}{3}dX_{[PQ]M}\wedge A^P\wedge A^Q - \tfrac{1}{2}Q_{(NM)}{}^E A^N\wedge B_E$$

$$-L_{MNP}A^N\wedge\left(dA^P + \tfrac{3}{8}X_{[RS]}{}^P A^R\wedge A^S\right) + \tfrac{1}{8}Q_{NP}{}^E T_{E\,QM}A^N\wedge A^P\wedge A^Q$$

$$-d(F_M - G_M) - X_{[MN]}{}^P A^N\wedge(F_P - G_P) + \tfrac{1}{2}\mathfrak{D}\vartheta_M{}^A\wedge B_A + \tfrac{1}{2}Q_{MP}{}^E C_E$$

$$\star\frac{\delta S}{\delta B_A} = \vartheta^{PA}(F_P - G_P) + Q^{AB}B_B - \mathfrak{D}\vartheta_M{}^A\wedge A^M - \tfrac{1}{2}Q_{NP}{}^A A^N\wedge A^P\,,$$

$$\tfrac{1}{2}\frac{\delta S}{\delta\vartheta_M{}^A} = (G_A{}^M - \tfrac{1}{2}\star\partial V/\partial\vartheta_M{}^A) - A^M\wedge(H_A + \tfrac{1}{2}\star j_A)$$

$$+\tfrac{1}{2}T_{A\,NP}A^M\wedge A^N\wedge(F^P - G^{\P}) - (F^M - G^M)\wedge B_A\,,$$

$$\frac{\delta S}{\delta D_{AB}} = Q_{AB}\,,\qquad \frac{\delta S}{\delta D_E{}^{NP}} = Q_{NP}{}^E\,,\qquad \frac{\delta S}{\delta D^{MNP}} = L_{MNP}\,.$$

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- ⋆ What happens in higher dimensions? (work in progress)