Supersymmetric non-Abelian monopoles and black holes in N=2,d=4 Super-Einstein-Yang-Mills Theories

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Based on 0712.1530 and 0806.1477.

Work done in collaboration with *M. Hübscher, P. Meessen and S. Vaulà* (IFT UAM/CSIC, Madrid)

Plan of the Talk:

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Now it is natural to ask what happens in the gauged theories. There are several possible gaugings in $N = 2, d = 4$ theories and we are going to work with a specific class that we call $N = 2, d = 4$ Super-Einstein-Yang-Mills theories. Let's describe these theories.

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$N = 2, d = 4$ SEYM theories do not include hypermultiplets.

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These transformations are not independent due to $\mathcal{N}_{\Lambda\Sigma}$. Furthermore, ordinary isometries are not symmetries of the full theory: **The isometries must preserve the Kähler, Hodge and special Kähler structures**.

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   (b) The group G includes an $SU(2)$ factor and acts on the spinors as a local $U(1) \times SU(2)$ R-symmetry via $SU(2)$ Fayet-Iliopoulos terms. (Work in progress).

The action of the bosonic fields of $N = 2, d = 4$ SEYM theories takes the form

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and $k_\Lambda{}^i(Z)$ are the holomorphic Killing vectors of the special-Kähler metric $\mathcal{G}_{ij^*}$ and $\mathcal{P}_\Lambda$ is the *momentum map*

$$k_\Lambda{}^i\mathcal{G}_{ij^*} = i\partial_{j^*}\mathcal{P}_\Lambda \,.$$

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1. We have to find, for all these theories, all the bosonic field configurations $e^a{}_\mu(x), A^\Lambda{}_\mu(x), Z^i(x)$ that admit Killing spinors first.

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1. We have to find, for all these theories, all the bosonic field configurations $e^a{}_\mu(x), A^\Lambda{}_\mu(x), Z^i(x)$ that admit Killing spinors first.
2. We have to impose the equations of motion to the supersymmetric configurations to determine which ones are supersymmetric solutions. This task is specially simple due to the existence of *Killing Spinor Identities* (KSIs) relating the off-shell e.o.m.s of supersymmetric configurations (Kallosh & O. `hep-th/9306085`, Bellorín & O. `hep-th/0501246`).

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In $N = 2, d = 4$ SEYM theories the Killing spinor equations

$$\delta_\epsilon \psi_{I\,\mu} = \mathfrak{D}_\mu \epsilon_I + \varepsilon_{IJ} T^+{}_{\mu\nu}\gamma^\nu \epsilon^J = \mathbf{0}\,, \quad \text{with } \mathfrak{D}_\mu \epsilon_I \equiv \{\nabla_\mu + \tfrac{i}{2}(\mathcal{Q}_\mu + g A^\Lambda{}_\mu \mathcal{P}_\Lambda)\}\epsilon_I\,,$$

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We have followed the method pioneered by Gauntlett et al. in hep-th/0209114 and have found the following...

General results

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The timelike class contains very interesting non-Abelian generalizations of the Abelian black-hole solutions.

We are going to focus on this case.

Our results for the timelike case can be summarized in the following

RECIPE:

☞ Find a set of Yang-Mills fields $\tilde{A}^{\Lambda}{}_{m}$ and functions $\mathcal{I}^{\Lambda}$ in $\mathbb{R}^3$ satisfying

$$\tfrac{1}{2}\,\epsilon_{xyz}\;\tilde{F}^{\Lambda}{}_{\underline{xy}} \;=\; -\tfrac{1}{\sqrt{2}}\;\tilde{\mathfrak{D}}_{\underline{z}}\mathcal{I}^{\Lambda}\;,$$

which is the Bogomol'nyi equation satisfied by known magnetic monopole solutions.

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☞ Use the above solution to find a solution of

$$\tilde{\mathfrak{D}}_m\tilde{\mathfrak{D}}_m\mathcal{I}_{\Lambda} = \tfrac{1}{2}g^2\,\left[f_{\Lambda(\Sigma}{}^{\Gamma}f_{\Delta)\Gamma}{}^{\Omega}\,\mathcal{I}^{\Sigma}\mathcal{I}^{\Delta}\right]\,\mathcal{I}_{\Omega}\,,$$

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$$\mathcal{I}_{\Lambda} \propto \mathcal{I}^{\Lambda}\,.$$

The real symplectic vector $\mathcal{I} = (\mathcal{I}^{\Lambda}, \mathcal{I}_{\Lambda})$ determines completely the solution. The physical fields $g_{\mu\nu}$, $A^{\Lambda}{}_{\mu}$, Z^{i} are derived from them as follows:

☞ Solve the *stabilization equations* to find $\mathcal{R}^\Lambda$ and $\mathcal{R}_\Lambda$. **N.B.**:

$$\begin{aligned} \mathcal{I}^\Lambda &\equiv \Im\mathrm{m}(\mathcal{L}^\Lambda/X)\,, & \mathcal{I}_\Lambda &\equiv \Im\mathrm{m}(\mathcal{M}_\Lambda/X)\,, \\ \mathcal{R}^\Lambda &\equiv \Re\mathrm{e}(\mathcal{L}^\Lambda/X)\,, & \mathcal{R}_\Lambda &\equiv \Re\mathrm{e}(\mathcal{M}_\Lambda/X)\,. \end{aligned}$$

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☞ and compute

$$2|X|^2 = \langle\, \mathcal{R} \mid \mathcal{I}\,\rangle^{-1}\,.$$

☞ The physical gauge field is given by

$$A^{\Lambda}{}_{\mu} dx^{\mu} = -\sqrt{2}|X|^{2}\mathcal{R}^{\Lambda}(dt + \hat{\omega}) + \tilde{A}^{\Lambda}{}_{\underline{x}} dx^{x} \, ,$$

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Let us now study some of the simplest examples (Hübscher, Meessen, O., Vaulà `arXiv/0712.1530`).

$SO(3)$ Examples:

Let us consider $N = 2$ EYM systems containing an $SO(3)$ gauge group, with indices $a = 1, 2, 3$.

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$$\mathcal{I}^a = \mathcal{I}\, n^a\,, \qquad A^a{}_m = \Phi\, \varepsilon_{mb}{}^a\, n^b\,, \qquad n^a \equiv x^a/r\,, \quad r \equiv \sqrt{x^b x^b}\,.$$

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A 2-parameter (μ and ρ) family of solutions is given by

$$\mathcal{I}(r) = \frac{\sqrt{2}\mu}{g}\mathsf{H}_\rho(\mu r) \,, \quad \mathsf{H}_\rho(r) = \coth(r+\rho) - \frac{1}{r} \,,$$

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The two most interesting cases are $\rho = 0, \infty$.

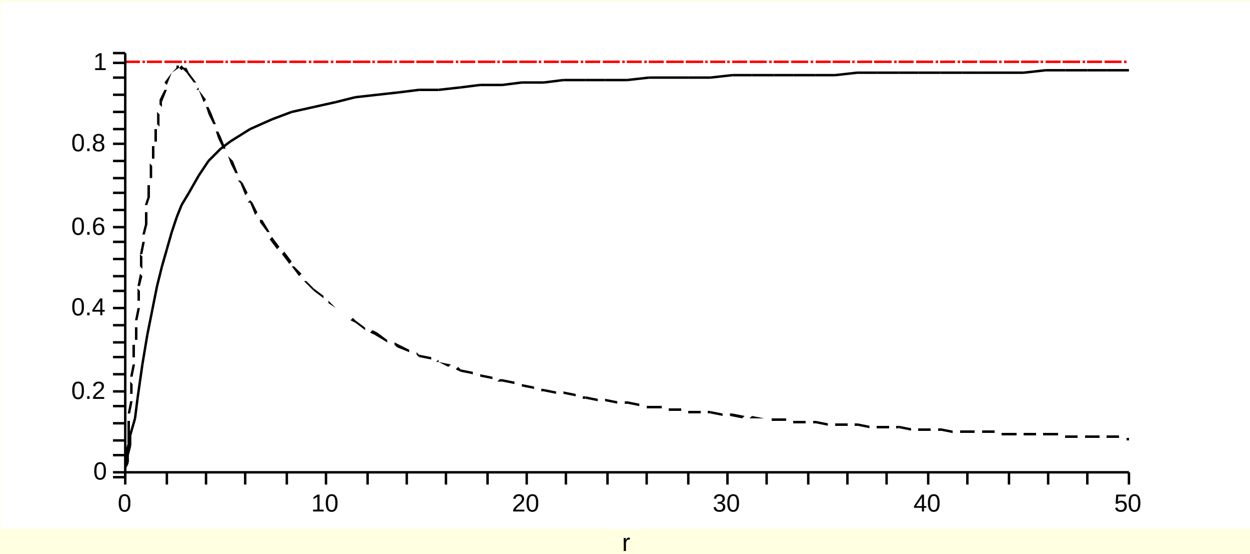
4 – 't Hooft-Polyakov Monopoles

The $\rho = 0$ solution can be written in the form

$$A^a{}_m = \varepsilon_{mb}{}^a\, n^b\, \frac{\mu}{g}\, \mathsf{G}_0(\mu r)\,, \quad \mathsf{G}_0(r) = \frac{1}{r} - \frac{1}{\sinh r}\,,$$

$$\mathcal{I}^a = \frac{\sqrt{2}\mu}{g}\, \mathsf{H}_0(\mu r)\, n^a\,, \quad \mathsf{H}_0(r) = \coth r - \frac{1}{r}\,.$$

The profiles of the functions G and H are



$\mathcal{I}^a$ is regular at $r = 0$ for $\rho = 0$, and describes the 't Hooft-Polyakov monopole.

5 – Black Hedgehogs

In the limit $\rho \to \infty$ we find the "black hedgehog" solution

$$\mathcal{I}^a = -\sqrt{2}\left(\mathcal{I}_\infty + \frac{1}{gr}\right) n^a \,,$$

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The possible existence of an event horizon covering the singularity at $r = 0$ has to be studied in specific models.

Before finding $\mathcal{R}$ and $|X|$ we have to find the $\mathcal{I}_a$s solving

$$\mathfrak{D}_m \mathfrak{D}_m \mathcal{I}_\Lambda \;=\; \tfrac{1}{2} g^2 \; \left[f_{\Lambda(\Sigma}{}^{\Gamma} f_{\Delta)\Gamma}{}^{\Omega} \, \mathcal{I}^{\Sigma} \mathcal{I}^{\Delta} \right] \, \mathcal{I}_\Omega \; ,$$

and solve the staticity constraint

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where $\mathcal{J}$ is an arbitrary constant.

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If we split the index Λ into an a-index and an u-index labeling the ungauged directions, the staticity constraint only acts non-trivially on the ungauged part:

$$\mathcal{I}_u\,d\mathcal{I}^u \;-\; \mathcal{I}^u\,d\mathcal{I}_u \;+\; \mathcal{I}_a\,\mathfrak{D}\mathcal{I}^a \;-\; \mathcal{I}^a\,\mathfrak{D}\mathcal{I}_a \;=\; \mathcal{I}_u\,d\mathcal{I}^u \;-\; \mathcal{I}^u\,d\mathcal{I}_u \;=\; 0\;,$$

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which we can solve as in the Abelian case or just set to zero.

This determines completely the family of solutions but, in order to find explicit expressions for $\mathcal{R}$ and $|X|$ and the spacetime metric we must solve the stabilization equations which depend on the specific model considered.

Metrics

For simplicity let us consider a $\overline{\mathbb{CP}}^3$ model whose prepotential reads

$$\mathcal{F} \;=\; \frac{i}{4}\,\eta_{\Lambda\Sigma}\;\mathcal{X}^{\Lambda}\;\mathcal{X}^{\Sigma}\,, \qquad \eta \;=\; \mathrm{diag}\,(\,-\;,\;[+]^{n}\;)\;.$$

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With the hedgehog Ansatz $\mathcal{I}^{a\,2} = \mathcal{I}^2$ and $SU(2)$ effectively reduces to a $U(1)$ in the metric!

Metrics

For simplicity let us consider a $\overline{\mathbb{CP}}^3$ model whose prepotential reads

$$\mathcal{F} \;=\; \tfrac{i}{4}\,\eta_{\Lambda\Sigma}\,\mathcal{X}^{\Lambda}\,\mathcal{X}^{\Sigma}\,, \qquad \eta \;=\; \mathrm{diag}\,(\,-\;,\;[+]^{n}\;)\;.$$

The Kähler potential is

$$e^{-\mathcal{K}} \;=\; 1\;-\;|Z|^2\,,\Rightarrow\;\;|Z|^2 < 1\,.$$

The stabilization equations are solved by

$$\mathcal{R}_{\Lambda} \;=\; -\tfrac{1}{2}\eta_{\Lambda\Sigma}\,\mathcal{I}^{\Sigma} \quad , \quad \mathcal{R}^{\Lambda} \;=\; 2\eta^{\Lambda\Sigma}\,\mathcal{I}_{\Sigma}\;,$$

and the metric function is given by

$$-g_{rr} = \frac{1}{2|X|^2} \;=\; -\tfrac{1}{2}\,\mathcal{I}^{\Lambda}\eta_{\Lambda\Sigma}\mathcal{I}^{\Sigma}\;-\;2\,\mathcal{I}_{\Lambda}\eta^{\Lambda\Sigma}\mathcal{I}_{\Sigma}\;=\;\tfrac{1}{2}\left[\mathcal{I}^{0\,2}\;-\;\mathcal{I}^{a\,2}\;+\;4\mathcal{I}_0{}^2\;-\;4\mathcal{I}_a{}^2\right]\;.$$

With the hedgehog Ansatz $\mathcal{I}^{a\,2} = \mathcal{I}^2$ and $SU(2)$ effectively reduces to a $U(1)$ in the metric! For black holes with finite entropy (attractor) we need at least two $U(1)$s. However, since $\mathcal{I}^a$ is bound in the monopole, we do not need $\mathcal{I}^0, \mathcal{I}_0$ and we can set them to constants.

Normalizing to have asymptotic flatness, we get, for the monopole

$$-g_{rr} = 1 \; + \; \mu^2 \left[\frac{1}{g^2} \; + \; \mathcal{J}^2\right] \; \left[1 \; - \; \mathsf{H}^2(\mu r)\right] \; ,$$

which is completely regular and describes an object of mass

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To embed the black hedgehog into this model and get a regular solution ($|Z|^2 < 1$) we need non-trivial $\mathcal{I}^0$ or $\mathcal{I}_0$. The conditions for regularity are the same as in an standard, Abelian $U(1) \times U(1)$ black hole of this model:

$$\mathsf{M} \ = \ \mathcal{I}^0_\infty p^0 + \mathcal{I}_{0\infty} q_0 - 2\mu \left[1/g^2 \ + \ \mathcal{J}^2\right] > 0 \, ,$$

$$\frac{A}{4\pi} \ = \ \tfrac{1}{2}[(p^0)^2 + 4(q_0)^2] - 2\frac{\mu^2}{g^2} \left[1/g^2 \ + \ \mathcal{J}^2\right] > 0 \, ,$$

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How does the attractor mechanism work in this solution?

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They have been shown (P. Meessen `arXiv:0803.0684`) to be regular black holes *with no asymptotic charges* just like the Bartnik-McKinnon one, but stable and given in a fully analytic form.

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- There is still much work to do to classify all the possible supersymmetric solutions....